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ABSTRACT

The theme of the meeting reported in this document was "Research in Mathematics Education from an Epistemological Perspective." In addition to the plenary session and reaction papers the conference proceedings contain papers grouped under the headings: (1) algebra; (2) computers and mathematics education; (3) early arithmetic (K-3); (4) geometry and spatial representation; (5) mathematical abilities; (6) pedagogical problems in the teaching of mathematics at the university level; (7) philosophy and perception of mathematics; (8) problem solving; (9) teacher training; (10) understanding, construction of meaning, cognitive levels, diagnosis; and (11) upper elementary and junior high school. The texts of the two plenary sessions and the two reaction papers appear in both English and French. In addition all English presentations offer a French abstract and vice versa. (PK)

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ACTES de la
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PME-NA

NORTH AMERICAN CHAPTER

of the

INTERNATIONAL GROUP

for the

PSYCHOLOGY OF MATHEMATICS EDUCATION

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FIFTH ANNUAL
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PME-NA

**NORTH AMERICAN CHAPTER
of the
INTERNATIONAL GROUP
for the
PSYCHOLOGY OF MATHEMATICS EDUCATION**

Vol. 1

**Edited by / Edité par
Jacques C. Bergeron
Nicolas Herscovics**

**MONTREAL
SEPT 29 — OCT 1, 1983**

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LAISSÉ AUX AUTEURS

PREFACE

The International Group for the Psychology of Mathematics Education (PME) was founded in 1976 at the third International Congress for Mathematics Education. The North American Chapter (PME-NA) was founded in 1979. Like its parent organization, the PME-NA sponsors an annual research conference bringing together psychologists, mathematics educators, mathematicians, philosophers, computer scientists, all interested in problems related to the learning and teaching of mathematics.

This year we have selected as a theme for the plenary sessions "Research in Mathematics Education from an Epistemological Perspective". Traditionnally epistemology is perceived as a branch of philosophy dealing with questions such as "What does it mean to know?", "How does knowledge develop?", "What are the conditions needed for its growth?". These are questions which seem fundamental for the science of education in general. But why choose it as a theme for a colloquium on research in mathematics education?

There are indications that the time is ripe to question the implicit philosophy underlying our past research. It seems that a significant part of the research community is now aware of the dead-end often achieved by a naive behaviorist approach towards mathematics education. For instance, such an approach has led to an algorithmic perception of mathematics, has viewed learning of mathematics as merely the acquisition of skills, and has considered teaching purely as a transmission of knowledge. On the other hand, a constructivist perspective stresses thinking and reasoning in the learning of mathematics, the acquisition of knowledge being perceived as a construction by the learner, and the teacher's role that of guiding him in his construction.

Two such opposite and diverging perspectives necessarily affect the kind of research questions one chooses and the methodologies one uses. The implications need to be discussed and debated thoroughly if we want to avoid two possible pitfalls, that of a researcher spending all his time trying to "solve rigorously" problems devoid of any pedago-

gical significance, or on the other hand trying to take into account so many variables that no possible experimentation can be construed.

In order to help open the debate, two plenary sessions and a panel discussion have been planned. The remaining papers have been grouped under the usual headings. To bring out the bilingual character of Canada, the texts of the two plenary papers and the two reaction papers appear in both English and French. In addition all English communications offer a French abstract and vice versa.

Jacques C. Bergeron
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PREFACE

Le groupe international PME (International Group for the Psychology of Mathematics Education) est né en 1976, lors du troisième Congrès International pour la didactique de la Mathématique (International Congress for Mathematics Education). Le chapitre nord-américain (PME-NA) a été fondé en 1979. A l'instar de l'organisme-mère, le PME-NA organise chaque année une conférence de recherche regroupant des psychologues, des formateurs de maîtres en didactique de la mathématique, des mathématiciens, des philosophes, des informaticiens, tous intéressés aux problèmes reliés à l'apprentissage et à l'enseignement de la mathématique.

Cette année nous avons choisi comme thème, pour les séances plénières, "La recherche en didactique de la mathématique dans une perspective épistémologique". Selon la tradition, l'épistémologie est perçue comme une branche de la philosophie traitant de questions telles que "Quel est le sens de connaître?", "Comment la connaissance se développe-t-elle?", "Quelles sont les conditions nécessaires à sa croissance?". Ce sont là des questions qui semblent fondamentales pour les sciences de l'Education en général. Mais pourquoi en faire le thème d'un colloque dédié à la recherche en didactique de la mathématique?

Il y a des indications que le temps est propice pour remettre en question la philosophie implicite sous-jacente à notre recherche passée. Il semble qu'un nombre assez important de chercheurs soient maintenant conscients du cul-de-sac dans lequel nous a souvent conduits une approche behavioriste naïve envers la didactique de la mathématique. Par exemple, une telle approche a mené vers une perception algorithmique de la mathématique, a réduit l'apprentissage de celle-ci à l'acquisition d'habiletés, et a considéré l'enseignement uniquement en termes d'une simple transmission de connaissances. Par contre, une perspective constructiviste met l'accent sur la pensée et le raisonnement dans l'apprentissage de la mathématique, l'acqui-

sition de la connaissance étant perçue comme une construction par l'apprenant, et le rôle de l'enseignant comme étant celui de guider l'élève dans sa construction.

Deux telles perspectives opposées et divergentes influencent nécessairement le type de questions de recherche que l'on se pose et le type de méthodologie que l'on adopte. Les implications qui en découlent ont besoin d'être discutées et débattues à fond, si nous voulons éviter deux des dangers possibles, soit celui du chercheur qui passe tout son temps à essayer de "résoudre de façon rigoureuse" des problèmes n'ayant aucune signification pédagogique, ou bien qui essaie de tenir compte de tant de variables qu'aucune expérimentation possible ne peut être envisagée.

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A. PLENARY PAPERS / CONFERENCES PLENIERES

WHY IS AN EPISTEMOLOGICAL PERSPECTIVE A NECESSITY FOR RESEARCH
IN MATHEMATICS EDUCATION

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Research in mathematics education studies the processes of transmission and appropriation of mathematical knowledge in order to understand them and eventually improve them.

The word transmission conveys the idea that mathematics education is a social function, with means and institutions, teachers and devices, books, pictures, films and computer-assisted-instruction programs, etc... It is an intentional activity which uses techniques and know-how and implies many decisions.

The word appropriation conveys two different ideas: one is that education is not necessarily successful and that the process of appropriation by students is an essential question; the other is that acquisition of a social knowledge like mathematics is not reducible to a process of spontaneous construction by children, adolescents and adults, even if one considers as essential a constructive approach to learning.

Imitation plays a very important part in education, but it very often fails when the know-how that is to be imitated and acquired is not meaningful for the student that is supposed to acquire it, or when it is too complex.

Meaning and complexity are distinct features that enlighten the problem of success and failure; they are not independent of each other, but not reducible to each other. For instance meaning may refer to social aspects of knowledge: some social groups may consider it as a "different" culture which has not much to do with the problems they meet in their own way of living. It may even be considered by some individuals as a cut from the family culture and, for this reason,

be rejected. Psycho-analysts have shown that mathematics may also be rejected because it appears as an expression of the social law and the father's law; they have also shown that mathematics may be loved for the very same reasons, and because they represent a secure world with well-fixed and limited rules. It is a complicated landscape.

My intention is not to deal with this social and psychoanalytical approach, but to concentrate upon epistemological aspects of meaning. Epistemology usually refers to problems of groundings and you all know the controversy between intuitionism and logicism, also the problem of reducing arithmetic to logic, or the different views about formalization and axiomatization. It is my view that we should extend the scope of epistemology to a larger category of questions which are essential to study the growth of knowledge in children's and students' minds. Epistemology of mathematics should not be limited to the initial basis of the building; it also concerns doors and windows, flats, bath-rooms, stair cases and lifts. I propose the following wide formulation of the question.

"To which problem or problems, does a new concept, a new property of a concept, a new procedure, or a new representation, bring a solution or eventually a more powerful one ? "

This understanding of epistemology is central to the history of science and technology. It is also central to studying the acquisition of knowledge by students.

The meaning of mathematics is conveyed by practical and theoretical problems to be solved, provided they are real problems for students. In other words, it is essential to analyse problems and situations that students are faced with or should be faced with, because this is the only way to make mathematics appear as a functional answer to real problems, and mathematical knowledge as an adaptative response to the environment.

The problem of complexity is a very important one: It does not refer only to the amount of information to be processed at the same time; my view is that epistemological difficulties are still more important and cannot be reduced to problems of memory-capacity: there are structures, and procedures, and representations that are conceptually difficult. It is the most important theoretical challenge for us to point to these difficulties, to describe and analyse them, and to propose ways for helping students to overtake them. Of course, meaning is not independent of complexity, and a question can be meaningless for a student because he cannot even catch the level of complexity where that question lies.

Referring to levels of complexity might induce the idea that all we need is a good description of general stages of intelligence development. This is not my view either for two reasons:

- first, the hierarchy of mathematical competences does not follow a total order organization, as the theory of stages unfortunately suggests, but rather a partial order one: situations and problems that students master progressively, procedures and symbolic representations they use, from the age of 2 or 3 up to adulthood and professional training, are better described by a partial-order scheme in which one finds competences that do not rely upon each other, although they may all require a set of more primitive competences, and all be required for a set of more complex ones.
- second, mathematical competences, like many other competences, cannot be properly described if one does not refer them to the specific contents they are dealing with. I will give several examples in this conference, but I want to stress now that this is probably the main argument for an epistemological perspective. The main problem for children and students is to increase their power upon the world, become more and more able to operate upon it in an efficient way, and consequently conceptualize it properly. The process of conceptualization inevitably depends on the contents of knowledge, either physical, economical, biological, social or otherwise. Mathematization of reality plays a very important part in conceptualization, although not exclusive.

I would like to use only a few examples for the sake of clarity, but I also need a variety of them to illustrate a variety of epistemological questions. So my list of examples is a compromise between clarity and complexity:

- volume;
- additive structures and conceptions of number;
- rational numbers, quantities and dimensional analysis;
- space as support of different symbolic representations.

VOLUME

The concept of volume is a good example to start with, because it is both a physical and a mathematical concept. Mathematization involves geometry and several branches of mathematics. I will consider here only the problem of quantifying the occupied space, at the elementary and secondary school levels. The epistemology of quantification requests the full recognition of the problems which students can (or must) be faced with:

- compare two volumes (or more);
- evaluate one volume with the help of another volume (smaller or bigger);
- approximate one volume;
- evaluate one volume by combining information about lengths, areas, angles...
- compare volumes in relationship with the comparison of lengths and areas;
- coordinate unit systems, and so on.

Young children are faced with some of these problems at a very young age: compare the amount of drink in two glasses; compare two containers, eventually by counting the number of cups contained in each of them; realize that the number found depends on the size of the cup; approximate measures that cannot be represented by whole numbers...

These problems do not imply any other cognitive operations but those conveyed by a unidimensional conception of volume: volume is a

measure of space contained in containers and it can be compared, added and subtracted. The addition axiom of the theory of measure is essential:

$$\text{Measure (A} \cup \text{B)} = \text{Measure (A)} + \text{Measure (B)}$$

provided A and B have no common part.

But it does not imply any relationship between volume and other spatial measures such as lengths and areas, except that there are some valid inferences like the following one:

If the shape of container A and container B is the same,
Height in A > height in B $\Rightarrow$ quantity in A > quantity in B.

Children don't need anything but a unidimensional conception of volume to say that:

- container X is bigger than container Y;
- container T is just as big as container X and container Y together;
- container W is 3 times as big as container Y;
- container T is two times and a half as big as container X plus a little;
- the measures (numbers) attached to these containers are not the same if you take a small cup or a larger one as a unit.

All these tasks can be studied with 5- to 11-year-old children, either in individual interviews or in the classroom. This study shows that the concept of volume cannot be characterized by the conservation of volume experiments only. Beside the important invariant that Piaget has characterized, one finds several relational invariants like those described above. The addition axiom for instance is a decisive step in the understanding of the unidimensional conception.

But this conception does not say much about the tridimensional conception of volume, as a product of measures: It does not say anything about the calculation of the volume of parallelepipeds, prisms, pyramids or spheres, nor about the calculation of the area of the

base knowing the volume and the height of a prism, nor about the measure of homothetic or similar shapes, nor about the relationship between the change of units of length and the change of units of volume.

There are situations in which you have to use these properties of the concept. The cognitive analysis of these situations is essential, so as to make the mathematical theory and the formulas really functional.

I do not have the time to develop all these considerations but I will mention a very important epistemological problem.

The metric system has many advantages. But it does not have only advantages. For instance it hides the combination of reasons that give its validity to the formula of the straight parallelepiped.

$$V = L \times W \times H$$

If this formula is true, it is because:

- volume is proportional to length when width and height are held constant;
- volume is proportional to width when length and height are held constant;
- volume is proportional to height when length and width are held constant;
- length, width and height are independant variables;
- the unit of volume has been chosen so as to be the volume of a parallelepiped of

$$V(1, 1, 1) = 1$$

If the last condition, which is specific to the metric system, were not satisfied, the volume would still be proportional to the product of all three measures and its most important properties would still be true. As a matter of fact, the essential first four reasons are not made explicit in mathematics education (in France anyway), although they are the most essential ones. One can trace the consequence of this under-estimation among 16-17 year-olds and even educated adults.

Psychology of mathematics education should certainly devote more effort to studying the relational invariants involved in the variety of problems (and cognitive tasks) related to a tridimensional conception of volume. My research group has started that study, but we are only at the beginning (Vergnaud, 1983). It is also a very interesting challenge to invent situations that can profitably help students to discover or understand all the properties involved.

There is a large variety of such situations: some of them may be meaningful for young children, some for older students; most of them can be profitably used in different grades. The appropriation of the concept of volume takes many years and goes across many steps and many different problems. It also interferes with the appropriation of additive structures, of proportion, of geometry, of other spatial measures. This complexity is not avoidable; mathematics education must deal with it, and psychology too.

ADDITIVE STRUCTURES AND CONCEPTIONS OF NUMBER

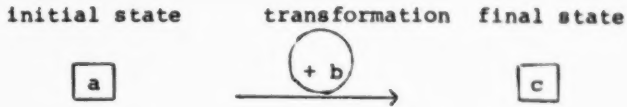
Psychologists and maths educators have already gathered many empirical and theoretical findings about that field. Some of them can be found in the book "Addition and Subtraction: a cognitive perspective" (Carpenter, Moser, Romberg, Eds, 1981). I would like to stress a few questions that seem to me very important from an epistemological point of view. I will consider only discrete quantities and whole numbers.

- How do students manage to make a unified theory of addition, starting from their primitive conception of addition as the increase of a quantity (Gelman, 1981) and discovering later the conception of addition as a commutative binary combination?

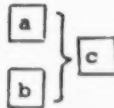
$$a + b = b + a = c$$

Whereas the teachers, the books (and the mathematicians) reinforce the binary internal combination model, the primitive conception of the child is best modeled by a unary external operation model in

which a directed number, either positive (addition) or negative (subtraction), operates upon a measure (initial state) and gives a measure (final state).



There is some discrepancy between this model, and the situations reflected by the combination of two measures which makes the binary model meaningful.



Whereas the addition axiom of the theory of measure concerns the binary model:

$$\text{cardinal } (A \cup B) = \text{card } (A) + \text{card } (B) \\ \text{provided } A \cap B \text{ is empty.}$$

the counting procedures of children are based on the unary model: see for instance the counting-on and the counting-down procedures.

Still children do make some synthesis. But the synthesis of what, at what moment, under what conditions ?

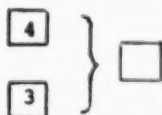
The complete synthesis cannot be made before measures, transformations, relationships, and abscissas are fused into one single powerful concept: the concept of directed number. Before that time, the cognitive tasks involved by the different classes of problems that children meet, can only permit partial synthesis and partial equivalences.

For instance there is some equivalence between a find-the-final-state problem



Fred had 4 marbles. He wins 3 marbles. How many does he have now ?

and a find-the-union problem



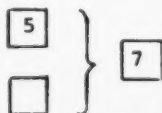
Susan has 4 sweets in her pocket and 3 sweets in her bedroom. How many altogether ?

or between a find-the-transformations problem (especially if the transformation is positive)



Gerald has 5 marbles before playing. He has now 7 marbles. What has happened during the game ?

and a find-the-complement problem



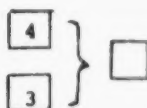
Kathleen counts 7 children at her birthday party. There are 5 girls. How many boys ?

But there is no such straight forward equivalence between a find-the-initial-state problem



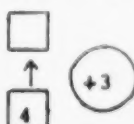
Fred has just lost 3 marbles. He has now 4 marbles. How many did he have before playing ?

and a find-the-union problem



(see Susan problem above)

or a find-the-compared-quantity problem



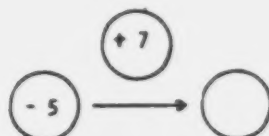
John has 3 sweets more than Carolyn. Carolyn has 4 sweets. How many does John have ?

The situation is still worse with more complex situations in which emerge either a combination of two transformations in succession, correctly modeled by a binary internal combination of directed numbers



Max has won 5 marbles at the first game ; lost 7 at the second game. What has he won or lost altogether ?

or the operation of a transformation upon a relationship (a comparison or a debt for instance), which is usually conceived as a unary operation, but can also be represented by a binary combination of directed numbers.



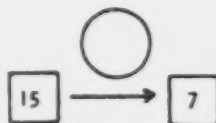
Stephanie owes 5 francs to her brother. She gives him 7 francs. What is the new situation ?

There may be several partial syntheses among all these situation-patterns. It is very well known for instance that the combination of transformations is often handled by children either with the help of a state-transformation-state metaphor, or with the help of a binary-combination-of-measures analogy.

Example: Victor found 15 francs in the street; then, he bought some cakes. When he counts his money, he finds 7 francs more than he had before going out. How much did he spend on cakes?

This problem can be handled successfully with a state-transformation-state-model

he has 15 francs , he has now 7 francs



find the transformation

Such handling would not have been successful if Victor, instead of finding 7 francs more, had found 7 francs less. In that case, the state-transformation-state metaphor produces a wrong answer, and the right answer implies, in one way or another, the combination of directed numbers.

So the epistemological landscape is quite complex for additive structures. On the one hand, children meet directed numbers at a very early age, as direct transformations, and later on as comparisons. On the other hand numbers are conceived as quantities, more than transformations or relationships. Natural numbers reflect this last conception. But when they solve problems, students often use a transformational conception of addition and subtraction, which is completely mis-represented by equalities or by Euler-Venn diagrams (Vergnaud, 1983). And finally all these different ingredients will have to be synthesized into the directed-number concept, natural numbers being identified with positive numbers, relationships with transformations, measures with positive transformations and relationships. At that moment the minus sign for instance will have to represent indifferently a direct negative transformation, the inversion of a positive one, the difference of two sets, of two states, of two transformations, a negative relationship or a negative abscissa.

I will consider later the problem of abscissas but I have said enough to show that there is no hope of making a complete cognitive analysis of these problems with the kind of general approach that psychologists usually use: logical instruments, information-processing, psycholinguistics, attention, and memory. The description of the mathematical contents of knowledge is, at this point, essential. Most operations of thinking involved in solving addition and subtraction problems can be described by theorems. As children and students are not necessarily able to make them explicit, although they use them, I have called them theorems-in-action.

RATIONAL NUMBERS, QUANTITIES AND DIMENSIONAL ANALYSIS

It is clear, from what has just been said, that the concept of number cannot come only from the synthesis of the concept of class and the concept of order as Piaget has sometimes said. Addition and subtraction are essential. If we did not have to measure discrete sets and other magnitudes, and if we did not have to add or subtract these

measures, there would not be anything like the concept of number. But numbers can also be multiplied and divided. What sort of problems do multiplication and division refer to?

They refer essentially to problems of proportion: simple proportion between two variables, double proportion between three variables: one variable being proportional to each one of the other two variables, independently. Multiple proportion of one variable to any of N other variables is an important structure in physics, and other domains; too little attention is paid to it, although it is a profound model for mathematizing reality. I have developed elsewhere my analysis of multiplicative structures. All I wish to do now is point to the problem of building the set $\mathbb{Q}$ of rational numbers.

Rational numbers come from fractions but there are at least two revolutions between the primitive conception of a fraction as a part-whole relationship, and the conception of a ratio bigger than one between two quantities or magnitudes of different kinds.

As a matter of fact, 7- or 8-year-olds have some understanding of the part-whole relationship, both for discrete and continuous quantities, for simple values like $1/2$ or $1/4$. One revolution consists in extending the concept to any part-whole value, and then to part-part ratios (i.e., disjoint quantities) and fractions bigger than 1. Another revolution consists in extending the concept to ratios between magnitudes of different kinds: $3/4$ dollar/cake, $7/5$ km/minute.

These two revolutions are tightly connected with multiplicative structures, in the sense that part-whole and part-part ratios connecting magnitudes of the same kind are scalar ratios, whereas ratios between magnitudes of different kinds have a functional aspect

$$\frac{f(x)}{x} = a$$

Extracting a functional ratio requires a higher-level abstraction than extracting a scalar ratio. Inverting it is also more difficult.

But there is another big epistemological problem with rational numbers: it comes from the fact that addition (and subtraction) of rationals inherits its meaning from fractions conceived as fractional quantities, whereas multiplication (and division) of rationals is more meaningful within the frame of fractional operators.

So the concept of rational number is the synthesis of many relationships and cognitive operations. Not only does it come from fractions and ratios, but it must also get rid of the magnitude-aspects that have made it meaningful, to become a pure number, on which the properties of magnitudes and the properties of operators are indifferently applied.

SPACE AS SUPPORT OF DIFFERENT SYMBOLIC REPRESENTATIONS

Space is probably the very first domain in which the feeling of necessity accompanies inferences made by children and there are probably many consequences of this.

Some of these consequences concern the problem of symbolic representation.

If symbolic representations help children in understanding relationships, it is necessarily because these relationships are well represented, and because the properties of these relationships can be better shown and analysed.

When representing magnitudes by lengths or by areas to understand order relationships, sums and differences, one uses the properties of spatial magnitudes to exhibit the properties of other magnitudes.

When using Euler-Venn representations for intersections, unions, and complements, one uses topological properties of space to show properties of classes.

I will develop a little more the number line problem, as I have

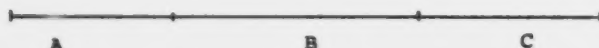
worked on it. Children do not understand the representation of numbers by the directed line as easily as one can imagine. Because 7- or 8-year-olds are already able to reproduce the graduation of their rule upon a line, teachers may have the illusion that they understand the number line properties. But which properties ?

In an experiment that some of you know because I have reported it in Berkeley, I have shown that 11-13-year-old students may have two different ideas: either they use the order properties of space (from left to right, from top to bottom...) to represent order relationships of magnitudes, ages, dates of birth; or they use the measures, refusing the principle of inclusion (no common part).

order

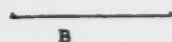
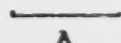


measure

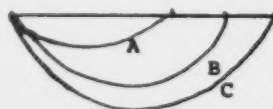


Among 60 categories of protocols, many of them just illustrate variations upon these two ideas.

The principle of inclusion of the different segments (signifiers) representing the different magnitudes is progressively overtaken by the following means:

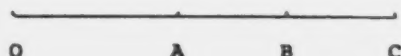


OR



As all segments have a common extremity, it is then possible, but it is not a straightforward idea, to identify the different segments with their second extremity.

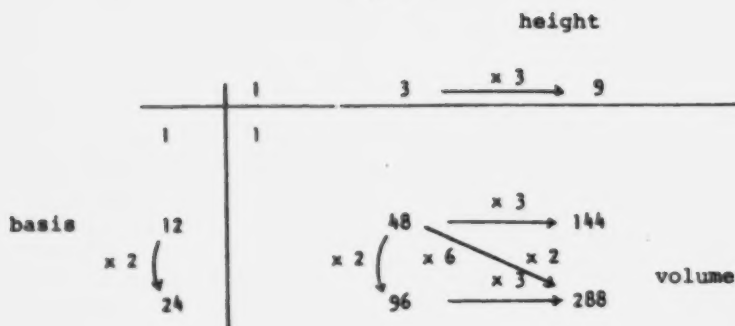
Then, and only then, is it possible to make the synthesis between the order properties of dots and the length properties of segments. The concept of origin accompanies this synthesis



Space as a signifier is then used to represent both order and magnitudes.

But this achievement does not yet give a status to intervals between dots which either represent differences between magnitudes or durations between events. This is especially visible when you need to subdivide these intervals. The status of intervals is not fixed for many students at the age of 13 or 14 years.

If I shift to another kind of symbolic representations like simple proportion tables and double proportion tables, one can observe that the projective properties of space (same direction, orthogonal direction, orthogonal direction, parallel columns and lines, parallel arrows and so on...) are used to represent different kinds of magnitudes and different properties of the linear and bilinear function. Let me give you just one example with the representation of the volume of prisms in a double proportion table.



This table exhibits clearly the isomorphic properties of the bilinear proportion. If you double the basis, the volume is also doubled. If you multiply the height by three, the volume is also multiplied by 3. If you do both, the volume is multiplied by 6.

In a simple proportion table, you can also represent differently the isomorphic properties and the constant coefficient properties of the linear function; you can also represent the composition of function and scalar operators.

This would not be possible if mathematical properties of space were not used in isomorphic correspondence with properties of magnitudes. The representational role of space relies upon these epistemological considerations.

CONCLUSION

Epistemology is first of all concerned with a problem of functionality. By this I mean that mathematical concepts, mathematical procedures, and mathematical representations are answers to problems that we must identify and analyze to understand how students deal with them and eventually discover or understand these answers.

To do the job, psychologists must be aware of the mathematical contents, and mathematicians must be aware of psychology. But the sort of cognitive psychology required by the analysis of the operations of thinking that take place, gradually, in students' minds, depends very much on the mathematical content itself.

It is essential to refer concepts to the situations in which they appear as an operational answer, but one discovers that the set of these situations is usually large (think of addition and subtraction for instance) and that the properties of the concept used in each class of situations may be different. The symbolic representations are also different and their function depends upon the properties to be exhibited.

I consider a concept as a triplet of three sets: $\text{Concept} = (S, I, \mathcal{Y})$

S: set of situations that make the concept meaningful;

I: set of invariants that constitute the concept (different properties, different levels);

$\mathcal{Y}$: set of symbolic representations that can be used to represent these properties and the situations.

S is the referent, I the signified, $\mathcal{Y}$ the signifier.

The core of cognition is in the operational invariants that students gradually discover or appropriate; these invariants can usually be described as theorems (very simple or very complex) which are not necessarily explicit. I call them "theorems in action", because it is essential to analyse them in mathematical terms.

But the discovery and appropriation of invariants take place in specific situations. So it is very important to analyse situations, classify them, experiment with them. Conceptions depend on situations met. Didactics must analyze the conditions in which conceptions depend on situations, and eventually change in facing new situations. This is what Brousseau has called the theory of didactic situations.

But the picture would not be complete if one did not see that symbols and symbolic representations play a very important part in identifying and analyzing the properties of the concepts at work. The analysis of the isomorphic properties of signifiers and signified is inescapable.

The full analysis of concepts, situations, and symbols cannot be made for each single property, or for each single situation. A situation cannot usually be analyzed with one single concept and a concept does not usually refer to one single situation. This is the reason why it is necessary to study conceptual fields, i.e., large sets of situations for the handling of which different interconnected concepts, procedures, and representations are necessary.

Conceptual fields usually develop over a long period of time.

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POURQUOI UNE PERSPECTIVE EPISTEMOLOGIQUE EST-ELLE NECESSAIRE
POUR LA RECHERCHE SUR L'ENSEIGNEMENT DES MATHÉMATIQUES?

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La recherche sur l'enseignement des mathématiques étudie les processus de transmission et d'appropriation du savoir mathématique, dans le but de les comprendre et éventuellement de les améliorer.

Le terme transmission véhicule l'idée que l'enseignement des mathématiques est une fonction sociale, qui dispose de moyens et d'institutions, de professeurs, de livres, de moyens audio-visuels et de programmes d'enseignement assisté par ordinateur... c'est une activité intentionnelle qui utilise des techniques et des savoir-faire et qui implique de nombreuses décisions.

Le terme appropriation véhicule deux idées distinctes: la première est que l'enseignement ne porte pas nécessairement ses fruits et que le processus d'appropriation par les élèves est une question essentielle; la seconde est que l'acquisition d'un savoir social comme les mathématiques n'est pas réductible à un processus de construction spontanée par les enfants, les adolescents et les adultes, même si l'on estime essentielle une approche constructiviste de l'apprentissage.

L'imitation joue un grand rôle dans l'éducation, mais elle échoue souvent lorsque le savoir-faire qu'il s'agit d'imiter et d'acquérir n'a pas de signification pour l'élève qui est censé l'acquérir, ou lorsqu'il est trop complexe.

Signification et complexité sont des aspects distincts, qui éclairent le problème de la réussite et de l'échec. Ils ne sont pas indépendants l'un de l'autre, mais ils ne sont pas réductibles l'un à l'autre. Par exemple, le problème de la signification peut renvoyer à des

aspects sociaux du savoir: certains groupes sociaux peuvent le considérer comme une culture "différente" qui n'a pas grand' chose à voir avec les problèmes que ces groupes sociaux rencontrent dans leur propre vie. Le savoir peut même être considéré par certains individus comme une véritable coupure à l'égard de la culture familiale et, pour cette raison, provoquer un phénomène de rejet. Les psychologues ont montré que les mathématiques peuvent également être rejetées parce qu'elles sont vécues comme une expression de la loi sociale et de la loi du père; ils ont aussi montré qu'elles peuvent être aimées pour des raisons tout à fait identiques, et parce qu'elles représentent un monde sûr, avec des règles bien fixées et limitées. Le paysage est donc complexe.

Mon intention n'est pas de traiter cette approche sociale et psychanalytique, mais de concentrer mon attention sur les aspects épistémologiques de la signification. L'épistémologie renvoie habituellement à des problèmes de fondement et vous connaissez tous les controverses entre l'intuitionnisme et le logicisme, ou encore le problème de la réduction de l'arithmétique à la logique, ou encore les vues différentes concernant la formalisation et l'axiomatisation. Mon opinion est que nous devons étendre le champ de la réflexion épistémologique à une catégorie plus large de questions, qui sont essentielles pour l'étude du développement des connaissances chez les enfants et les adolescents. L'épistémologie ne doit pas être réduite à la base de l'édifice des mathématiques; elle concerne également les portes et les fenêtres, les appartements et les salles de bain, les escaliers et les ascenseurs. Je propose une formulation ouverte.

"A quel(s) problème(s) un concept nouveau, une nouvelle propriété d'un concept, une nouvelle procédure, une nouvelle représentation apportent-ils une solution, ou éventuellement une solution plus puissante que les solutions existantes?"

Cette manière de voir l'épistémologie est essentielle pour l'histoire des sciences et des techniques. Elle est également essentielle pour

l'étude de la formation des connaissances chez les élèves.

La signification des mathématiques vient des problèmes pratiques et théoriques à résoudre, à condition qu'ils soient considérés comme des problèmes réels par les élèves. En d'autres termes, il est essentiel d'analyser les problèmes et les situations auxquels les élèves sont confrontés ou devraient être confrontés, parce que c'est la seule manière de faire apparaître les mathématiques comme une réponse fonctionnelle à des problèmes réels, et le savoir mathématique comme une réponse adaptative à l'environnement.

Le problème de la complexité est très important: la complexité ne se réduit pas à la quantité d'information à traiter en même temps; mon opinion est que les difficultés épistémologiques sont encore plus importantes et ne peuvent pas être réduites à une question de capacité de la mémoire immédiate: il y a des structures, des procédures et des représentations qui sont conceptuellement difficiles. C'est actuellement l'enjeu le plus important pour nous, que de repérer ces difficultés, de les décrire et de les analyser, et de proposer des moyens pour aider les élèves à les surmonter. Bien entendu, la signification n'est pas indépendante de la complexité; et une question peut n'avoir aucune signification pour un élève, justement parce qu'il ne peut pas saisir le niveau de complexité auquel se situe cette question.

Cette référence à la complexité pourrait induire l'idée que ce qu'il nous faut, c'est une bonne description des stades généraux du développement de l'intelligence. Ce n'est pourtant pas mon opinion, et ceci pour deux raisons distinctes.

- premièrement, la hiérarchie des compétences mathématiques n'obéit pas à une structure d'ordre total comme la théorie des stades le suggère malencontreusement, mais plutôt à une structure d'ordre partiel: les situations et les problèmes maîtrisés progressivement par les élèves, les procédures et les représentations symboliques qu'ils

utilisent, à partir de l'âge de 2 ou 3 ans jusqu'à l'âge adulte, y compris au cours de la formation professionnelle, sont mieux décrites par un schéma d'ordre partiel, dans lequel on peut trouver des compétences qui ne dépendent pas l'une de l'autre, alors qu'elles s'appuient toutes sur un même ensemble de compétences plus primitives, et qu'elles sont toutes nécessaires pour des compétences plus élaborées;

- deuxièmement, les compétences mathématiques, comme beaucoup d'autres compétences, ne peuvent pas être décrites correctement sans qu'il soit fait référence au contenu de connaissance spécifique sur lequel elles portent. Je fournirai plusieurs exemples au cours de cette conférence, mais je veux souligner maintenant qu'il s'agit là du principal argument à l'appui d'une perspective épistémologique. Le principal problème pour les enfants et les élèves, c'est d'accroître leur pouvoir sur le monde, c'est-à-dire d'accroître leurs capacités à agir dessus de manière efficace; ils doivent pour cela le conceptualiser correctement. Ce processus de conceptualisation dépend inévitablement du contenu du savoir en jeu, qu'il s'agisse d'un savoir de type physique, économique, biologique, social ou autre. La mathématisation de la réalité joue un rôle important dans ce processus de conceptualisation, encore qu'il ne soit pas exclusif.

Je n'utiliserai que peu d'exemples, par souci de clarté, mais j'ai cependant besoin d'une certaine variété si je veux en même temps illustrer une variété de questions épistémologiques. La liste d'exemples est donc le résultat d'un compromis:

- le volume;
- les structures additives et les conceptions du nombre;
- les nombres rationnels, les grandeurs, et l'analyse dimensionnelle;
- l'espace comme support de différentes représentations symboliques.

LE VOLUME

Le concept de volume est un premier exemple intéressant parce que c'est à la fois un concept physique et un concept mathématique. Sa mathématisation implique la géométrie et plusieurs branches des

mathématiques. Je ne considérerai ici que le problème de la quantification de l'espace occupé, aux niveaux de l'école élémentaire et de l'école secondaire. L'épistémologie de la quantification requiert la reconnaissance pleine et entière des problèmes auxquels les étudiants peuvent (ou doivent) être confrontés:

- comparer deux volumes (ou plus);
- évaluer un volume à l'aide d'un autre volume (plus petit ou plus grand);
- évaluer un volume par approximation;
- évaluer un volume en combinant des informations concernant les longueurs, les surfaces, les angles;
- comparer des volumes en liaison avec la comparaison des longueurs et des surfaces;
- coordonner des systèmes d'unités, etc...

Les jeunes enfants sont confrontés à certains de ces problèmes à un âge très précoce: comparer la quantité de boisson dans deux verres, comparer deux récipients éventuellement en comptant le nombre de tasses que peut contenir chacun d'eux, prendre conscience que le nombre de tasses dépend de la taille de la tasse, trouver une approximation des mesures qui ne peuvent pas être représentées par des nombres entiers.

Ces problèmes n'impliquent pas d'autre opération cognitive que celles contenues dans une conception unidimensionnelle du volume: le volume est une mesure de l'espace contenu dans les récipients et il peut faire l'objet de comparaisons, d'additions et de soustractions. L'axiome additif de la théorie de la mesure est essentiel:

$$\text{mesure } (A \parallel B) = \text{mesure } (A) + \text{mesure } (B)$$

pourvu que A et B n'aient pas de partie commune.

mais il n'implique aucune relation entre le volume et les autres mesures spatiales telles que les longueurs et les aires, à l'exception

de quelques inférences valides comme la suivante:

Si la forme d'un récipient A et d'un récipient B est
la même, alors
hauteur dans A > hauteur dans B $\Rightarrow$ contenu de A > contenu de B

Les enfants n'ont besoin de rien d'autre qu'une conception unidimensionnelle du volume pour déclarer que:

- le récipient X est plus grand que le récipient Y;
- le récipient T est juste aussi grand que les récipients X et Y ensemble;
- le récipient W est trois fois plus grand que le récipient Y;
- le récipient T est deux fois et demie plus grand que le récipient X, plus un petit peu;
- les mesures (nombres) associées à ces récipients ne sont pas les mêmes si l'on prend une petite tasse ou une plus grande comme unité.

Toutes ces tâches peuvent être étudiées avec des enfants de 5 à 11 ans, soit en entretiens individuels, soit en classe. Cette étude montre que le concept de volume ne peut pas être caractérisé par la seule conservation du volume. A côté de cet important invariant, que Piaget a décrit, on trouve plusieurs invariants relationnels comme ceux décrits ci-dessus. L'axiome d'addition, par exemple, est un pas décisif dans la compréhension de la conception unidimensionnelle du volume.

Mais cette conception est muette à l'égard de la conception tridimensionnelle du volume comme produit de mesures: elle ne dit rien sur le calcul du volume des parallélépipèdes, des prismes, des pyramides ou de la sphère, ni sur le calcul de l'aire de la base d'un prisme connaissant son volume et sa hauteur, ni sur la mesure des formes homothétiques ou semblables, ni sur la relation entre le changement d'unité de longueur et le changement d'unité de volume.

Il y a des situations dans lesquelles il faut utiliser ces propriétés du concept. L'analyse cognitive de ces situations est essentielle,

afin de rendre la théorie mathématique et les formules véritablement fonctionnelles.

Je n'ai pas le temps de développer toutes ces considérations mais je mentionnerai un très important problème épistémologique.

Le système métrique a beaucoup d'avantages. Mais il n'a pas que des avantages. Par exemple, il cache la combinaison de raisons qui donnent sa validité à la formule du parallélépipède droit

$$V = L \times l \times h$$

Si cette formule est vraie, c'est parce que:

- le volume est proportionnel à la longueur quand la largeur et la hauteur sont constantes;
- le volume est proportionnel à la largeur quand la longueur et la hauteur sont constantes;
- le volume est proportionnel à la hauteur quand la longueur et la largeur sont constantes;
- longueur, largeur et hauteur sont des variables indépendantes;
- l'unité de volume a été choisie de telle manière que cette unité soit le volume d'un parallélépipède de longueur = 1, de largeur = 1, de hauteur = 1.

En d'autres termes, comme variable trilineaire, le volume satisfait la propriété

$$V(1,1,1) = 1$$

Si cette dernière condition, qui est spécifique du système métrique, n'était pas satisfaite, le volume serait encore proportionnel au produit des trois mesures et ses propriétés les plus importantes continueraient d'être vraies. De fait, les quatre premières raisons ne sont guère explicites dans l'enseignement des mathématiques (en France en tous cas), bien qu'elles soient les plus essentielles. On peut détecter les conséquences de cette sous-estimation parmi les élèves de 16-17 ans et même parmi les adultes cultivés.

La psychologie de l'enseignement des mathématiques devrait consacrer un effort plus grand à l'étude des invariants relationnels impliqués dans la variété de situations (et de tâches cognitives) qui sont associées à une conception tridimensionnelle du volume. Mon groupe de recherche a commencé ce travail, mais nous en sommes seulement au début (Vergnaud, 1983). C'est aussi un challenge intéressant d'inventer des situations qui puissent aider les élèves à découvrir et à comprendre toutes les propriétés impliquées par les différentes conceptions du volume.

Il y a une grande variété de telles situations: certaines d'entre elles ont du sens pour les jeunes enfants, d'autres pour les plus âgés; la plupart d'entre elles peuvent être utilisées avec profit à différents niveaux scolaires. L'appropriation du concept de volume prend beaucoup d'années et traverse beaucoup d'étapes et de problèmes différents. Elle interfère avec l'appropriation des structures additives, des proportions, de la géométrie, des autres mesures spatiales. Cette complexité n'est pas évitable; l'enseignement des mathématiques doit s'en occuper, la psychologie également.

LES STRUCTURES ADDITIVES ET LES CONCEPTIONS DU NOMBRE

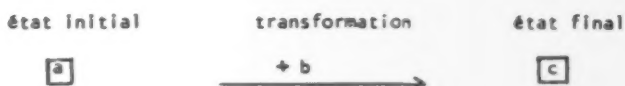
Les psychologues et les enseignants de la mathématique ont déjà rassemblé beaucoup de résultats empiriques et théoriques sur ce domaine.

Certains d'entre eux peuvent être trouvés dans le livre "Addition and Subtraction: a cognitive perspective" (Carpenter, Moser, Romberg, (Eds), 1981). Je voudrais souligner quelques questions qui m'apparaissent importantes d'un point de vue épistémologique. Je ne considérerai, dans ce qui suit, que des quantités discrètes et des nombres entiers.

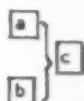
- Comment les élèves font-ils pour former une théorie unifiée de l'addition, en partant de leur conception primitive de l'addition comme accroissement d'une quantité (Gelman, 1981), et en découvrant un peu plus tard la conception de l'addition comme une composition binaire

commutative $a + b = b + a = c$?

Alors que les maîtres, les manuels (et les mathématiciens) renforcent le modèle de la composition interne binaire, la conception primitive de l'enfant est mieux modélisée par un modèle d'opération externe unaire, dans lequel un nombre relatif, soit positif (addition), soit négatif (soustraction), opère sur une mesure (état initial) et donne une mesure (état final).



Il y a un écart entre ce modèle et les situations qui impliquent la combinaison de deux mesures et qui rendent significatif le modèle binaire



Alors que l'axiome additif de la théorie de la mesure concerne le modèle binaire

$$\text{cardinal } (A \cup B) = \text{card } (A) + \text{card } (B)$$

pourvu que $A \cap B$ soit vide

les procédures de comptage des enfants privilégient le modèle unaire: voir par exemple les procédures de comptage en avant et en arrière (counting on et counting down).

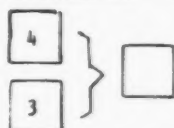
Pourtant les enfants font bel et bien une sorte de synthèse. Mais la synthèse de quoi, à quel moment, sous quelles conditions ?

La synthèse complète ne peut pas être faite avant que les mesures, les transformations, les relations et les abscisses ne fusionnent en un même concept puissant: le concept de nombre relatif. Avant cela, les tâches cognitives impliquées par les différentes classes de problèmes que rencontrent les enfants, peuvent seulement permettre des synthèses et des équivalences partielles.

Par exemple, il y a quelque équivalence entre un problème de recherche de l'état final

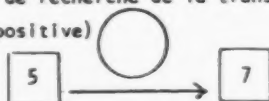


Fred avait 4 billes. Il gagne 3 billes. Combien de billes a-t-il maintenant? et un problème de recherche de l'union

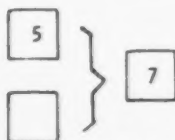


Susan a 4 bonbons dans sa poche et 3 bonbons dans sa chambre. Combien de bonbons a-t-elle en tout?

- ou entre un problème de recherche de la transformation (surtout si la transformation est positive)

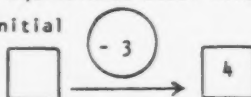


Gérald avait 5 billes avant de jouer. Il a maintenant 7 billes. Que s'est-il passé pendant la partie? et un problème de recherche du complément

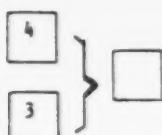


Kathleen compte 7 enfants pour son anniversaire. Il y a 5 filles. Combien y a-t-il de garçons?

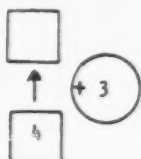
Mais il n'existe pas une équivalence aussi directe entre un problème de recherche de l'état initial



Fred vient juste de perdre 3 billes. Il a maintenant 4 billes. Combien de billes avait-il avant de jouer? et un problème de recherche de l'union



(voir le problème de Susan ci-dessus), ou un problème de recherche de la quantité comparée



John a 3 billes de plus que Carolyn. Carolyn a 4 bonbons. Combien John a-t-il de bonbons?

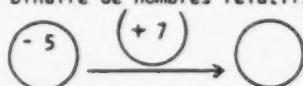
La situation est encore pire avec des situations plus complexes dans lesquelles se fait jour

- soit la combinaison de deux transformations successives, correctement modélisée par une combinaison interne binaire de nombres relatifs,



Max a gagné 5 billes à la première partie, perdu 7 à la seconde partie. Qu'a-t-il gagné ou perdu en tout?

- soit encore l'opération d'une transformation sur une relation (une comparaison ou une dette par exemple), laquelle est habituellement pensée comme une opération unaire, mais peut également être représentée par une combinaison binaire de nombres relatifs



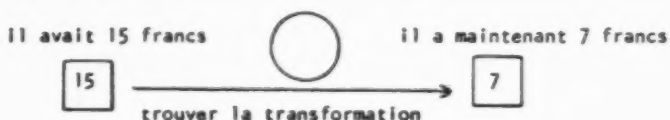
Stéphanie doit 5 francs à son frère. Elle lui donne 7 francs. Quelle est la situation nouvelle?

Il peut y avoir plusieurs synthèses partielles parmi tous ces patterns de situations. Il est bien connu par exemple que la combinaison de transformations est souvent traitée par les enfants, soit à l'aide

d'une métaphore de type état-transformation-état, soit à l'aide d'une composition de mesures binaires.

Exemple: Victor a trouvé 15 francs dans la rue, ensuite il a acheté des gâteaux. Quand il compte son argent, il trouve 7 francs de plus que ce qu'il avait avant de sortir. Combien a-t-il dépensé en gâteaux?

Ce problème peut être traité avec succès à l'aide d'un modèle de type état-transformation-état



Un tel traitement n'aurait pas conduit à la réussite si Victor, au lieu de trouver 7 francs de plus, avait trouvé 7 francs de moins. Dans ce cas, le modèle état-transformation-état produit une réponse fausse, et la réponse juste implique, d'une manière ou d'une autre, la combinaison de nombres relatifs.

Ainsi le paysage épistémologique est complexe pour les structures additives. D'une part, les enfants rencontrent les nombres relatifs à un âge précoce, comme transformations directes, et plus tard comme comparaisons. D'autre part, les nombres sont conçus comme des quantités, bien plus que comme des transformations ou des relations. Les nombres naturels sont l'image de cette dernière conception. Mais quand ils résolvent des problèmes, les élèves utilisent le plus souvent une conception transformationnelle de l'addition et de la soustraction, qui n'est pas adéquatement représentée par les égalités dans $\mathbb{N}$, ni par les diagrammes d'Euler-Venn (Vergnaud, 1983). En dernier ressort tous ces ingrédients différents devront être synthétisés dans le concept de nombre relatif, les nombres naturels étant alors identifiés aux nombres positifs, grâce à l'identification des relations aux transformations, des mesures aux transformations et aux relations positives. A ce moment le signe moins par exemple devra

représenter indifféremment une transformation directe négative, l'inversion d'une transformation positive, la différence de deux ensembles, de deux états, de deux transformations, ou encore une relation ou une abscisse négative.

Je reviendrai plus loin sur le problème des abscisses, mais j'en ai dit assez pour montrer qu'il n'y a aucun espoir de faire une analyse cognitive complète des structures additives avec le type d'approche générale que les psychologues utilisent habituellement: les instruments logiques, le traitement de l'information, la psycholinguistique, l'attention et la mémoire. La description du contenu mathématique du savoir est, à ce point, essentielle. La plupart des opérations de pensée impliquées dans les problèmes d'addition et de soustraction peuvent être décrites par des théorèmes. Comme les enfants et les élèves ne sont pas nécessairement capables de les expliciter, bien qu'ils les utilisent dans la solution de problème, je les ai appelés "théorèmes-en-acte".

LES NOMBRES RATIONNELS, LES GRANDEURS ET L'ANALYSE DIMENSIONNELLE

Il est bien clair, à partir de ce qui vient d'être dit, que le concept de nombre ne peut pas venir seulement de la synthèse du concept de classe et du concept d'ordre comme Piaget l'a parfois déclaré. L'addition et la soustraction sont essentielles. Si nous n'avions pas à mesurer les ensembles discrets et les autres grandeurs, et si nous n'avions pas à ajouter ou à soustraire des mesures, il n'y aurait rien de tel que le concept de nombre. Mais les nombres peuvent aussi être multipliés et divisés. A quelles sortes de problèmes la multiplication et la division réfèrent-elles?

Essentiellement à des problèmes de proportion: soit à la proportion simple entre deux variables, soit à la double proportion entre trois variables: l'une d'entre elles étant proportionnelle à chacune des deux autres, de manière indépendante. La proportion multiple d'une variable par rapport à l'une quelconque de n autres variables est une structure importante en physique, et dans d'autres domaines; une

attention trop faible lui est consacrée, bien qu'il s'agisse d'un modèle très profond pour la mathématisation du réel. J'ai développé ailleurs mon analyse des structures multiplicatives. Tout ce que je souhaite faire maintenant c'est concentrer mon regard sur le problème de la construction de l'ensemble des nombres rationnels.

Le concept de nombre rationnel est issu du concept de fraction, mais il y a au moins deux révolutions entre la conception primitive de fraction comme relation partie-tout, et la conception d'un rapport plus grand que 1 entre des quantités ou des grandeurs de nature différente.

Il est de fait que les enfants de 7 ou 8 ans ont une certaine compréhension de la relation partie-tout, pour les quantités discrètes et continues, et pour des valeurs simples comme $1/2$ ou $1/4$. Une première révolution consiste à étendre le concept à n'importe quelle relation partie-tout, puis à n'importe quel rapport partie-partie (rapport entre parties disjointes): par conséquent à des fractions plus grandes que 1. Une seconde révolution consiste à étendre le concept à des rapports entre grandeurs de nature différente: $3/4$ dollars par gâteau, $7/5$ km/minute.

Ces deux révolutions sont étroitement associées aux structures multiplicatives, en ce sens que les rapports partie-tout et les rapports partie-partie entre grandeurs de même nature sont des rapports scalaires, tandis que les rapports entre grandeurs de nature différente ont un caractère fonctionnel $\frac{f(x)}{x} = a$

L'extraction d'un rapport de type "fonction", requiert un niveau d'abstraction plus élevé que l'extraction d'un rapport "scalaire". Son inversion est également plus difficile.

Mais il y a encore un autre gros problème épistémologique avec les nombres rationnels: l'addition (et la soustraction) des nombres rationnels puise sa signification dans les fractions conçues comme des quantités fractionnaires, tandis que la multiplication (et la division)

des nombres rationnels tire davantage sa signification du concept d'opérateurs fractionnaires.

Ainsi, le concept de nombre rationnel est la synthèse de nombreuses relations et opérations cognitives. Non seulement il sort des concepts de fraction et de rapport, mais il doit en même temps se débarrasser des propriétés qui lui ont donné sa signification, pour devenir un nombre pur, sur lequel les propriétés des grandeurs et celles des opérations peuvent être indifféremment appliquées.

L'ESPACE COMME SUPPORT DE DIFFERENTES REPRESENTATIONS SYMBOLIQUES

L'espace est probablement le domaine premier dans lequel se forge le sentiment de nécessité qui accompagne les inférences faites par les enfants, et il s'ensuit probablement de nombreuses conséquences.

Certaines de ces conséquences concernent le problème des représentations symboliques. Si les représentations symboliques aident les enfants à comprendre les relations, c'est nécessairement parce que ces relations sont adéquatement représentées, et parce que leurs propriétés peuvent être mieux mises en évidence et analysées.

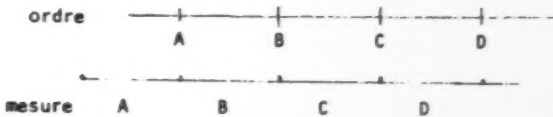
Quand on représente des grandeurs par des longueurs ou des surfaces pour favoriser la compréhension des relations d'ordre, des sommes ou des différences, on utilise les propriétés des grandeurs spatiales pour mettre en évidence les propriétés d'autres grandeurs.

Quand on utilise des représentations d'Euler-Venn pour les intersections, les unions et les compléments, on utilise les propriétés topologiques de l'espace pour mettre en évidence des propriétés des classes d'objets.

Je voudrais développer un peu plus le problème de la droite numérique, parce que j'ai étudié ce problème avec un certain détail. Les enfants ne comprennent pas la représentation des nombres sur la droite orientée, aussi facilement qu'on peut l'imaginer. Parce que les enfants

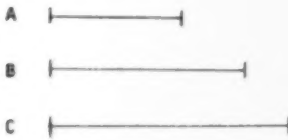
de 7 ou 8 ans sont déjà capables de reproduire la graduation de leur règle sur une droite, les maîtres peuvent avoir l'illusion qu'ils comprennent les propriétés de la droite numérique. Mais quelles propriétés?

Dans une expérience que certains d'entre nous connaissent, parce que je l'ai rapportée à Berkeley, j'ai montré que les élèves de 11-13 ans peuvent avoir deux idées différentes: ou bien ils utilisent les propriétés d'ordre de l'espace (de la gauche vers la droite, du haut vers le bas...) pour représenter les relations d'ordre entre grandeurs, âges, dates de naissance; ou bien ils utilisent le caractère mesurable de l'espace pour représenter des mesures absolues, en refusant tout principe d'inclusion des mesures entre elles (pas de partie commune).

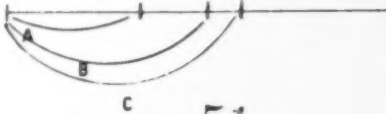


Parmi 60 catégories de protocoles, beaucoup d'entre eux illustrent des variations autour de ces deux idées.

Le principe d'inclusion des différents segments (signifiants) qui représentent les différentes grandeurs est progressivement surmonté par les moyens suivants:

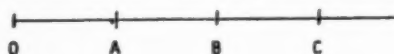


ou encore



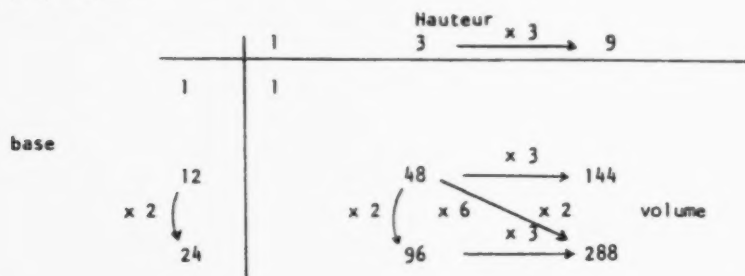
Comme tous les segments ont une extrémité commune, il est alors possible, mais ce n'est pas une idée qui va de soi, d'identifier les différents segments avec leur seconde extrémité.

Alors, et alors seulement, il est possible de faire la synthèse entre les propriétés d'ordre des points et les propriétés mesurables des segments (longueur). Le concept d'origine accompagne cette synthèse.



L'espace comme signifiant est alors utilisé pour représenter à la fois l'ordre et les grandeurs mesurables. Mais cette construction élaborée ne suffit pas à donner tout leur statut aux intervalles entre points, qui représentent soit des différences entre grandeurs, soit des durées entre événements. Ceci est particulièrement visible quand vous avez besoin de subdiviser ces intervalles. Le statut des intervalles n'est pas totalement fixé pour beaucoup d'élèves à l'âge de 13 ou 14 ans.

Si maintenant je passe à un autre type de représentation comme les tableaux de proportion simple ou les tableaux de proportion multiple, on peut observer que les propriétés projectives de l'espace (même direction, direction orthogonale, colonnes et lignes parallèles, flèches parallèles, etc...) sont utilisées pour représenter différents types de grandeurs et différentes propriétés de la fonction linéaire ou de la fonction bilinéaire. Permettez-moi de vous donner un seul exemple avec la représentation du volume du prisme dans une table de double proportion.



Le tableau montre clairement les propriétés d'isomorphisme de la fonction bilinéaire. Si l'on double la base, le volume est également doublé. Si l'on multiplie la hauteur par 3, le volume est également multiplié par 3. Si l'on fait les deux à la fois, le volume est multiplié par 6.

Dans un tableau de proportion simple, on peut également représenter de manière différente les propriétés d'isomorphisme de la fonction linéaire et les propriétés qui concernent le coefficient constant; on peut aussi représenter la composition des fonctions et des opérations scalaires.

Ceci ne serait pas possible si les propriétés mathématiques de l'espace n'étaient pas utilisées et mises en correspondance avec les propriétés des grandeurs en jeu. Le rôle de l'espace dans les représentations symboliques repose donc sur des considérations épistémologiques.

CONCLUSION

L'épistémologie est en premier lieu concernée par un problème de fonctionnalité. Je veux dire par là que les concepts mathématiques, les procédures mathématiques et les représentations mathématiques sont des réponses à des problèmes qu'il nous faut identifier et analyser pour comprendre comment les élèves traitent ces problèmes et découvrent ou comprennent ces réponses.

Pour entreprendre ce travail, les psychologues doivent être bien informés sur les contenus mathématiques, et les mathématiciens bien informés sur la psychologie. Mais le type de psychologie cognitive requis pour l'analyse des opérations de pensée qui interviennent au cours du développement des connaissances chez les élèves, dépend beaucoup des contenus mathématiques eux-mêmes.

Il est essentiel d'établir une référence entre les concepts et les situations auxquelles ils apportent une réponse opérationnelle, mais on découvre rapidement que ces situations forment un ensemble assez

large et différencié (pensons à l'addition et à la soustraction par exemple), et que les propriétés du concept utilisé dans chaque classe de situations peuvent être différentes. Les représentations symboliques également sont différentes, et leur fonction dépend des propriétés à mettre en évidence.

Je considère qu'un concept est un triplet de trois ensembles

$$\text{Concept} = (S, I, R)$$

S: ensemble de situations qui donnent son sens au concept.

I: ensemble des invariants constitutifs du concept (différentes propriétés, différents niveaux,...)

R: ensemble des représentations symboliques qui peuvent être utilisées pour représenter ces propriétés et les situations.

S est le référent, I le signifié, R le signifiant.

Le noyau de la connaissance se trouve dans les invariants opératoires que les élèves découvrent ou s'approprient progressivement; ces invariants peuvent habituellement être décrits comme des théorèmes (très simples ou très complexes) qui ne sont pas nécessairement explicites. Je les appelle des "théorèmes-en-acte" parce qu'il est essentiel de les analyser dans des termes mathématiques.

Mais la découverte et l'appropriation de ces invariants se fait dans des situations spécifiques. Aussi est-il très important d'analyser les situations, de les classer, de les expérimenter. Les conceptions dépendent des situations rencontrées. La didactique analyse les conditions dans lesquelles les conceptions dépendent des situations et éventuellement se modifient lorsqu'elles sont confrontées à des situations nouvelles. C'est à cela que Brousseau a donné le nom de "théorie des didactiques".

Mais le paysage ne serait pas complet si l'on ne voyait pas que les symboles et les représentations symboliques jouent un rôle important dans l'identification et l'analyse des propriétés des concepts en jeu.

L'analyse des isomorphismes entre signifiants et signifiés est incontournable.

Ces analyses des concepts, des situations et des symboles ne peut pas être faite pour chaque propriété singulière, ni pour chaque situation singulière. Une situation ne peut pas généralement être analysée avec un seul concept, et un concept ne renvoie pas généralement à un seul type de situations. C'est la raison pour laquelle il est nécessaire d'étudier des champs conceptuels, c'est-à-dire des ensembles larges de situations, pour le traitement desquelles sont nécessaires des concepts, des procédures, et des représentations en étroite interconnexion.

Les champs conceptuels font l'objet d'un développement cognitif qui s'étale généralement sur une longue période de temps.

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LEARNING AS A CONSTRUCTIVE ACTIVITY

Ernst von Glasersfeld, University of Georgia

The general topic I was given for this talk is "Research in Mathematics Education from an Epistemological Perspective". That sounds no more dangerous than so many other academic topics. But don't let the prosaic surface deceive you. To introduce epistemological considerations into a discussion of education has always been dynamite. Socrates did it, and he was promptly given hemlock. Giambattista Vico did it in the 18th century, and the philosophical establishment could not bury him fast enough. In our own time there was Jean Piaget. He really wanted to stay out of education but allowed himself to be drawn in -- and we know what has happened to his epistemology at the hands of interpreters and translators. It seems that to discuss education from an epistemological point of view was a sure way of committing intellectual suicide. Recently, however, the world of education may have begun to change. At least the particular discipline that is concerned with numbers, with arithmetic, and ultimately with mathematics, is manifesting symptoms that indicate the will to change.

The rapid shifts in the methods of mathematics education that have taken place in the last few decades -- from simplistic associationism to "New Math" and "Back to Basics" -- did not work the miracles that were expected of them. Their failure has created a mood that no longer fosters enthusiasm for new gimmicks. Today, I think it is fair to say, there is a more or less general disillusionment. This disillusionment is healthy and propitious because it pushes us closer to the point where we might be ready to review some of the fundamental presuppositions of the traditional theories of education. Among these presuppositions are our conception of teaching and learning and, most fundamental of all, the conception of what it is "to know"

Ten or fifteen years ago, it would have been all but inconceivable

to subject educators or educational researchers to a talk that purported to deal with a theory of knowledge. Educators were concerned with getting knowledge into the heads of their students, and educational researchers were concerned with finding better ways of doing it. There was, then, little if any uncertainty as to what the knowledge was that students should acquire, and there was no doubt at all that, in one way or another, knowledge could be transferred from a teacher to a student. The only question was, which might be the best way to implement that transfer -- and educational researchers with their criterion-referenced tests and their sophisticated statistical methods were going to provide the definitive answer.

Something, apparently, went wrong. Things did not work out as expected. Now there is disappointment, and this disappointment -- I want to emphasize this -- is not restricted to mathematics education but has come to involve teaching and the didactic methods in virtually all disciplines. To my knowledge, there is only one exception that forms a remarkable contrast: the teaching of physical and, especially, athletic skills. There is no cause for disappointment in that area. In those same ten or fifteen years in which the teaching of intellectual matters has somehow foundered, the teaching of skills such as tennis and skiing, pole jumping and javelin throwing has advanced quite literally by leaps and bounds. The contrast is not only spectacular but it is also revealing. I shall return to this phenomenon at a later point when, I hope, we will be able to consider an analogy which, at this moment, might seem utterly absurd.

If educational efforts are, indeed, failing, the presuppositions on which, implicitly or explicitly, these efforts have been founded must be questioned and it seems eminently reasonable to suggest, as did those who formulated the topic for this discussion, that we begin by inspecting the commodity that education claims to deal in, and that is "knowledge".

This paper is an attempt to do three things. First, I shall go back to what I consider the origin of the troubles we have had with the traditional conception of knowledge. This historical review will not only be sketchy, but it will also be quite biased, because I have rather strong views on the subject. However, considering the mess in which the theory of knowledge has been during the last fifty years in the "hard" sciences, my attempt will, I hope, not be deemed unjustified.

Second, I shall propose a conceptualization of "knowledge" that does not run into the same problem and that, moreover, provides another benefit in that it throws helpful light on the process of communication. As teachers, I said a moment ago, we are intent upon generating knowledge in students. That, after all, is what we are being paid for, and since the guided acquisition of knowledge, no matter how we look at it, seems predicated on a process of communication, we should take some interest in how that process might work. In my experience, this is an aspect that has not been given much thought. Educators have spent and are rightly spending much time and effort on curriculum. That is, they do their best to work out what to teach and the sequence in which it should be taught. The underlying process of linguistic communication, however, the process on which their teaching relies, is usually simply taken for granted. There has been a naive confidence in language and its efficacy. Although it does not take a good teacher very long to discover that saying things is not enough to "get them across", there is little if any theoretical insight into why linguistic communication does not do all it is supposed to do. The theory of knowledge which I am proposing, though it certainly does not solve all problems, makes this particular problem very clear.

Lastly, having provided what I would like to call a model of "knowing" that incorporates a specific view of the process of imparting knowledge, I shall briefly explore a way to apply that model to the one

thing all of us here are interested in: how to introduce children to the art, the mystery, and the marvelous satisfaction of numerical operations.

I

The nature of knowledge was a hotly debated problem as far back as the 6th century B.C. The debate has been more or less continuous, and while in many ways it has been colorful, it has been remarkably monotonous in one respect. The central problem has remained unsolved throughout, and the arguments that created the major difficulty at the beginning are the very same that today still preclude any settlement of the question.

The story begins with the first documents on epistemology that have come down to us, the so-called "fragments" of the Pre-Socratics. The ideas these men struggled with and tried to clarify must have arisen some time before them, but since we have no earlier written records, that background is extremely hazy. The Pre-Socratics, at any rate, exhibit a degree of sophistication that is unlikely to have been acquired in one or two generations. Fragmentary though they are, their pronouncements leave no doubt that towards the close of the 5th century B.C. the process of knowing had been conceptually framed in a relatively stable general scenario. By and large, the thinkers who concerned themselves with the cognizing activity tacitly accepted the scenario in which the knower and the things of which, or about which, he or she comes to know are, from the outset, separate and independent entities.

I want to stress that this dichotomy does not coincide with the split between the knowing subject and the subject's knowledge. That second dichotomy appears whenever an actor becomes aware of his or her own activity or when a thinker begins to think about his or her own thinking. That second problem of self-consciousness is not

identical with the problem of cognition. Though the two are related in that they intersect (e.g., in an analysis of reflective thought, which will enter our discussion later), I here want to deal only with the first. The Pre-Socratics, in any case, took for granted the human ability to be aware of knowing. What they began to wonder about was how it was possible that one could come to know the world. It is in this quest that the cognitive scenario they accepted and that has been perpetuated by almost all epistemologists after them, is of decisive importance. Once it was chosen as the basis for the construction of a theory of knowledge, that construction was saddled with a paradox. The paradox is inescapable and it has haunted philosophers incessantly in the 2500 years since then.

The reason why that particular cognitive scenario was adopted is very simple. It reflects the situation as it initially appears to any experienter. The question, how it comes about that we know anything, is not likely to be asked at the beginning of a prospective knower's development. A six-year-old who bicycles home from school would be a very peculiar six-year-old if she suddenly asked herself, "How on earth do I manage to find my way home?" or "What exactly happened when I learned to ride my bicycle?"

I am not suggesting that these are questions a six-year-old or, indeed, anyone should ask. I am merely saying that if we ever do ask them, it will be at a somewhat later age. The same goes for the question, "How is it that I can know what I do know?" Those who have felt such epistemological curiosity probably formulated their first relevant question in their middle teens or later. That is to say, they began to question their knowledge at a point in their cognitive career when they had already acquired an enormous amount of know-how and learning. Inevitably, nearly all they knew at that point was tacitly assumed to be knowledge about the environment, about the world in which they found themselves living. It is not surprising that this should be the case. Once one has learned to

manage things, there is no reason to suspect that they might not be what they seem.

If a person, whose knowledge has been growing and expanding over the years, then raises questions about how one comes to have all that knowledge, it seems reasonable to postulate at the beginning an experienced and totally ignorant knower, who comes into the world, much as an explorer might come into a terra incognita, with both the need and the will to discover what that world is like. The first if not the only tools that seem to be available for such a task are obviously the senses. Therefore, the senses are at once categorized as organs, or channels, through which the experiencer receives messages from the environment. On the basis of these messages, the experiencer then must, and apparently can, build up a "picture" of the world. In our contemporary jargon, this is often expressed by saying that the senses convey information which enables the experiencing subject to form a representation of the world. Usually this seems to work quite well. Occasionally, of course, the senses turn out to be somewhat deceptive, but by and large they work well enough for us to build up a modus vivendi. Provided we remain patient and flexible, we will continue to make adjustments, and as long as things work moderately well, there will be no need to question the over-all validity of whatever picture of the world we have built up.

The Pre-Socratics started out in this thoroughly normal way, but because there were some highly original thinkers among them, they came up with mutually incompatible pictures of the world.⁽¹⁾ Obviously, that was felt to be a problem and it led to two closely connected questions: One, how could anyone compose a picture of the world out of sensory messages and, two, how could one be certain that a particular picture of the world was "true"? Attempts to answer these questions soon ran into troubles, some of them so serious that they have not yet been overcome.

Here I want to focus on the second problem because it is inherent and unavoidable in the discoverer's scenario. If experience is the only contact a knower can have with the world, there is no way of comparing the products of experience with the reality from which whatever messages we receive are supposed to emanate. The question, how veridical the acquired knowledge might be, can therefore not be answered. To answer it, one would have to compare what one knows with what exists in the "real" world -- and to do that, one would have to know what "exists". The paradox, then, is this: to assess the truth of your knowledge you would have to know what you come to know before you come to know it.

The argument that the likeness or trustworthiness of a picture can be assessed only by looking at both the picture and what it is supposed to depict, was brought forth already at the time of the Pre-Socratics and it has been the mainstay of all scepticism ever since. The history of Western epistemology is the history of more or less imaginative attempts to circumvent it. None of these attempts was satisfactory. Plato's poetic genius almost succeeded in eliminating the dilemma by undercutting the role of experience. He placed the real reality into the world of ideas and turned sensory experience into a secondary affair, murky, unreliable, and ultimately irrelevant to the quest for truth. Since the world of ideas was accessible only to the thinking mind, this arrangement bred the notion of solipsism, the notion that there is no real world beyond the world the mind creates for itself.

Alternatively one could, as Descartes suggested, place one's faith in God and trust the divine maker not to have been so malicious as to have provided his creatures with deceptive senses.

Neither of the two alternatives provides a durable solution. Solipsism turns into absurdity whenever an idea we have conceived is shattered by experience. In fact, that is not a rare occurrence.

We are constantly reminded that the world we live in is not quite the world we would like and that there is, indeed, a hard and unforgiving "reality" with which we have to cope. On the other hand, the Cartesian notion boils down to a simple injunction to believe, and that does not satisfy the philosopher's need. If epistemology must be founded on the blind faith that our senses convey a true picture, it cannot accomplish what it sets out to do, namely provide a rational basis for the generation and assessment of knowledge. Actually, Descartes' injunction to trust our God-given senses merely shifts the problem. If the senses were thought to be trustworthy, the fact that we so often draw the wrong conclusion from their messages should show that there is a serious difficulty of interpretation; and if we cannot be sure how to interpret what the senses tell us, we again have to admit that we have no certain knowledge of the world and that the picture we come to have of it remains questionable.

The problem, as I suggested at the beginning, is intrinsic to the traditional scenario. It arises from the "iconic" conception of knowledge, a conception that requires a match or correspondence between the cognitive structures and what these structures are supposed to represent. Truth, in that conception, inevitably becomes the perfect match, the flawless representation. The moment we accept that scenario, we begin to feel the need to assess just how well our cognitive structures match what they are intended to represent. But that "reality" lies forever on the other side of our experiential interface. To make any such assessment of truth we should have to be able, as Hilary Putnam recently put it, to adopt a "God's eye view" (Putnam, 1982). Since we are not, and logically cannot be, in a position to have such a view of the "real" world and its presumed representation, there is no way out of the dilemma. What we need is a different scenario, a different conception of what it is "to know", a conception in which the goodness of knowledge is not predicated on likeness or representation.

The first explicit proposal of a different approach originated in those quarters that were most concerned with faith and its preservation. When, for the first time, the revolutionary notion that the Earth might not be the center of the universe seriously threatened the picture of the world which the church held to be unquestionable and sacred, it was the defenders of the faith who proposed an alternative scenario for the pursuit of scientific knowledge. In his preface to Copernicus' treatise De revolutionibus, Osiander (1627) suggested:

There is no need for these hypotheses to be true, or even to be at all like the truth; rather, one thing is sufficient for them -- that they yield calculations which agree with the observations.
(POPPER, 1968)

This introduces the notion of a second kind of knowledge, apart from faith and dogma, a knowledge that fits observations. It is knowledge that human reason derives from experience. It does not represent a picture of the "real" world but provides structure and organization to experience. As such it has an all-important function: it enables us to solve experiential problems.

In Descartes' time, this instrumentalist theory of knowledge was formulated and developed by Mersenne and Gassendi.⁽²⁾ It was then extended by Berkeley and Vico, given strong but unintended support by Hume and Kant; and at the end of the last century, it was applied to physics and science in general by Ernst Mach and to philosophy by Georg Simmel.⁽³⁾ It was not and still is not a theory popular with traditional philosophers. The idea that knowledge is good knowledge if and when it solves our problems is not acceptable as criterion to those who continue to hope that knowledge, ultimately, will at least approximate a true picture of the "real" world.

Karl Popper, who has given a lucid account of the beginnings of instrumentalism (Popper, 1968) has struggled hard to convince us that,

though reasonable, it is an unsatisfactory theory. As he reiterates in his latest work:

What we are seeking, in science, are true theories -- true statements, true descriptions of certain structural properties of the world we live in. These theories or systems of statements may have their instrumental use; yet what we are seeking in science is not so much usefulness as truth: approximations to truth; explanatory power, and the power of solving problems; and thus, understanding. (Popper, 1982)

This suggests that "descriptions", "explanations", and "understanding" can indeed capture aspects of "the world we live in". Whether we can or cannot agree with this statement will depend on how we define "the world we live in". There is no doubt that Popper intended an objective world, i.e., a ready-made world into which we are born and which, as explorers, we are supposed to get to know. This is the traditional realist view, and Popper does his best to defend it, in spite of all arguments one can hold up against it. The realists and the sceptics are once more in the familiar deadlock.

Yet, there is another possibility. "The world we live in" can be understood also as the world of our experience, the world as we see, hear, and feel it. This world does not consist of "objective facts" or "things-in-themselves" but of such invariants and constancies as we are able to compute on the basis of our individual experience. To adopt this reading, however, is tantamount to adopting a radically different scenario for the activity of knowing. From an explorer who is condemned to seek "structural properties" of an inaccessible reality, the experiencing organism now turns into a builder of cognitive structures intended to solve such problems as the organism perceives or conceives. Fifty years ago, Piaget characterized this scenario as neatly as one could wish: "Intelligence organizes the world by organizing itself"(Piaget, 1937). What determines the value of the conceptual structures is their experimental adequacy,

their goodness of fit with experience, their viability as means for the solving of problems, among which is, of course, the never-ending problem of consistent organization that we call understanding.

The world we live in, from the vantage point of this new perspective, is always and necessarily the world as we conceptualize it. "Facts", as Vico saw long ago, are made by us and our way of experiencing, rather than given by an independently existing objective world. But that does not mean that we can make them as we like. They are viable facts as long as they do not clash with experience, as long as they remain tenable in the sense that they continue to do what we expect them to do.

This view of knowledge, clearly, has serious consequences for our conceptualization of teaching and learning. Above all, it will shift the emphasis from the student's "correct" replication of what the teacher does, to the student's successful organization of his or her own experience. But before I expand on that I want to examine the wide-spread notion that knowledge is a commodity that can be communicated.

II

The way we usually think of "meaning" is conditioned by centuries of written language. We are inclined to think of the meaning of words in a text rather than of the meaning a speaker intends when he or she is uttering linguistic sounds. Written language and printed texts have a physical persistence. They lie on our desks or can be taken from shelves, they can be handled and read. When we understand what we read, we gain the impression that we have "grasped" the meaning of the printed words, and we come to believe that this meaning was in the words and that we extracted it like kernels out of their shells. We may even say that a particular meaning is the "content" of a word or of a text. This notion of words as containers in which

the writer or speaker "conveys" meaning to readers or listeners is extraordinarily strong and seems so natural that we are reluctant to question it. Yet, it is a misguided notion. To see this, we have to retrace our own steps and review how the meaning of words was acquired at the beginning of our linguistic career.

In order to attach any meaning to a word, a child must, first of all, learn to isolate that particular word as a recurrent sound pattern among the totality of available sensory signals. Next, she must isolate something else in her experiential field, something that recurs more or less regularly in conjunction with that sound pattern. Take an ordinary and relatively unproblematic word such as "apple". Let us assume that a child has come to recognize it as a recurrent item in her auditory experience. Let us further assume that the child already has a hunch that "apple" is the kind of sound pattern that should be associated with some other experiential item. Adults interested in the child's linguistic progress can, of course, help in that process of association by specific actions and reactions, and they will consider their "teaching" successful when the child has come to isolate in her experiential field something that enables her to respond in a way which they consider appropriate. When this has been achieved, when the appropriate association has been formed, there is yet another step the child must make before she can be said to have acquired the meaning of the word "apple". The child must learn to re-present to herself the designated compound of experiences whenever the word is uttered, even when none of the elements of that compound are actually present in her experiential field. That is to say, the child must acquire the ability to imagine or visualize, for instance, what she has associated with the word "apple" whenever she hears the sound pattern of that word.⁽⁴⁾

This analysis, detailed though it may seem, is still nothing but a gross summary of certain indispensable steps in a long procedure of interactions. In the present context, however, it should suffice

to justify the conclusion that the compound of experiential elements that constitutes the concept an individual has associated with a word cannot be anything but a compound of abstractions from that individual's own experience. For each one of us, then, the meaning of the word "apple" is an abstraction which he or she has made individually from whatever apple-experiences he or she has had in the past. That is to say, it is subjective in origin and resides in a subject's head, not in the word which, because of an association, has the power to call up, in each of us, our own subjective representation.

If you grant this inherent subjectivity of concepts and, therefore, of meaning, you are immediately up against a serious problem. If the meanings of words are, indeed, our own subjective construction, how can we possibly communicate? How could anyone be confident that the representations called up in the mind of the listener are at all like the representations the speaker had in mind when he or she uttered the particular words? This question goes to the very heart of the problem of communication. Unfortunately the general conception of communication was derived from and shaped by the notion of words as containers of meaning. If that notion is inadequate, so must be the general conception of communication.

The trouble stems from the mistaken assumption that, in order to communicate, the representations associated with the words that are used must be the same for all communicators. For communication to be considered satisfactory and to lead to what we call "understanding", it is quite sufficient that the communicators' representations be compatible in the sense that they do not manifestly clash with the situational context or the speaker's expectations.

A simple example may help to make this clear. Let us assume that, for the first time, Jimmy hears the word "mermaid". He asks what it means and is told that a mermaid is a creature with a woman's head and torso and the tail of a fish. Jimmy need not have met such

a creature in actual experience to imagine her. He can construct a representation out of familiar elements, provided he is somewhat familiar with and has established associations to 'woman', 'fish', and the other words used in the explanation. However, if Jimmy is not told that in mermaids the fish's tail replaces the women's legs, he may construct a composite that is a fish-tailed biped and, therefore, rather unlike the intended creature of the seas. Jimmy might then read stories about mermaids and take part in conversations about them for quite some time without having to adjust his image. In fact, his deviant notion of a mermaid's physique could be corrected only if he got into a situation where the image of a creature with legs as well as a fish's tail comes into explicit conflict with a picture or with what speakers of the language say about mermaids. That is, Jimmy would modify the concept that is his subjective meaning of the word only if some context forced him to do so.

Now, you may now ask, can a context force one to modify one's concepts? The question must be answered not only in a theory of communication but also in a theory of knowledge. The answer I am proposing is essentially the same in both.

The basic assumption is one that is familiar to you. Organisms live in a world of constraints. In order to survive, they must be "adapted" or, as I prefer to say, "viable". This means that they must be able to manage their living within the constraints of the world in which they live. This is a commonplace in the context of biology and evolution. In my view, the principle applies also to cognition -- with one important difference. On the biological level, we are concerned with species, i.e., with collections of organisms which, individually, cannot modify their biological make-up. But since they are not all the same, the species "adapts" simply because all those individuals that are not viable are eliminated and do not reproduce. On the cognitive level, we are concerned with individuals and specifically with their "knowledge" which, fortunately, is not

immutable and only rarely fatal. The cognitive organism tries to make sense of experience in order better to avoid clashing with the world's constraints. It can actively modify ways and means to achieve greater viability.

"To make sense" is the same activity and involves the same presuppositions whether the stuff we want to make sense of is experience in general or the particular kind of experience we call communication. The procedure is the same but the motivation, the reason why we want to make sense, may be different.

Let me begin with ordinary experience. No matter how one characterizes cognizing organisms, one of their salient features is that they are capable of learning. Basically, to have "learned" means to have drawn conclusions from experience and to act accordingly. To act accordingly, of course, implies that there are certain experiences which one would like to repeat rather than others which one would like to avoid. The expectation that any such control over experience can be gained must be founded on the assumptions that (1) some regularities can be detected in the experiential sequence and (2) future experience will at least to some extent conform to these regularities. These assumptions, as David Hume showed, are prerequisites for the inductive process and the knowledge that results from it.

In order to find regularities, we must segment our experience into separate pieces so that, after certain operations of recall and comparison, we may say of some of them that they recur. The segmenting and recalling, the assessing of similarities, and the decisions as to what is to be considered different, are all our doing. Yet, whenever some particular result of these activities turns out to be useful (in generating desirable or avoiding undesirable experiences) we quickly forget that we could have segmented, considered, and assessed otherwise. When a scheme has worked several times, we come to believe, as Piaget has remarked, that it could not be otherwise

and that we have discovered something about the real world. Actually we have merely found one viable way of organizing our experience. "To make sense" of a given collection of experiences, then, means to have organized them in a way that permits us to make more or less reliable predictions. In fact, it is almost universally the case that we interpret experience either in view of expectations or with a view to making predictions about experiences that are to come.

In contrast, "to make sense" of a piece of language does not usually involve the prediction of future non-linguistic experience. However, it does involve the forming of expectations concerning the remainder of the piece that we have not yet heard or read. These expectations concern words and concepts, not actions or other experiential events. The piece of language may, of course, be intended to express a prediction, e.g., "tomorrow it will rain", but the way in which that prediction is derived from the piece of language differs from the way in which it might be derived from, say, the observation of particular clouds in the sky. The difference comes out clearly when it is pointed out that, in order to make sense of the utterance "tomorrow it will rain" it is quite irrelevant whether or not there is any belief in the likelihood of rain. To "understand" the utterance, it is sufficient that we come up with a conceptual structure which, given our past experience with words and the way they are combined, fits the piece of language in hand. The fact that, when tomorrow comes, it doesn't rain, in no way invalidates the interpretation of the utterance. On the other hand, if the prediction made from an observation of the sky is not confirmed by actual rain, we have to conclude that there was something wrong with our interpretation of the clouds.

In spite of this difference between the interpretation of experience and the interpretation of language, the two have one important feature in common. Both rely on the use of conceptual material which the interpreter must already have. "Making sense", in both cases, means finding a way of fitting available conceptual elements into

a pattern that is circumscribed by specific constraints. In the one case, the constraints are inherent in the way in which we have come to segment and organize experience; in the other, the constraints are inherent in the way in which we have learned to use language. In neither case is it a matter of matching an original. If our interpretation of experience allows us to achieve our purpose, we are quite satisfied that we "know"; and if our interpretation of a communication is not countermanded by anything the communicator says or does, we are quite satisfied that we have "understood".

The process of understanding in the context of communication is analogous to the process of coming to know in the context of experience. In both cases, it is a matter of building up, out of available elements, conceptual structures that fit into such space as is left unencumbered by constraints. Though this is, of course a spatial metaphor, it illuminates the essential character of the notion of viability and it brings out another aspect that differentiates that notion from the traditional one of "truth": having constructed a viable path of action, a viable solution to an experiential problem, or a viable interpretation of a piece of language, there is never any reason to believe that this construction is the only one possible.

III

When I began the section on communication by talking about the concept of meaning, it must have become apparent that I am not a behaviorist. For about half a century behaviorists have worked hard to do away with "mentalistic" notions such as meaning, representation, and thought. It is up to future historians to assess just how much damage this mindless fashion has wrought. Where education is concerned, the damage was formidable. Since behaviorism is by no means extinct, damage continues to be done, and it is done in many ways. One common root, however, is the presumption that all that matters -- perhaps even all there is -- are observable stimuli and observable

responses. This presumption has been appallingly successful in wiping out the distinction between training and education.

As I hope to have shown in the preceding section, a child must learn more than just to respond "apple" to instantiations of actual apple experiences. If that were all she could do, her linguistic proficiency would remain equivalent to that of a well-trained parrot. For the bird and its trainer to have come so far is a remarkable achievement. For a human child it is a starting-point in the development of self-regulation, awareness, and rational control.

As mathematics educators, you know this better than most. To give correct answers to questions within the range of the multiplication table is no doubt a useful accomplishment, but it is, in itself, no demonstration of mathematical knowledge. Mathematical knowledge cannot be reduced to a stock of retrievable "facts" but concerns the ability to compute new results. To use Piaget's terms, it is operative rather than figurative. It is the product of reflection -- and while reflection as such is not observable, its products may be inferred from observable responses.

I am using "reflection" in the sense in which it was originally introduced by Locke, i.e., for the ability of the mind to observe its own operations. Operative knowledge, therefore, is not associative retrieval of a particular answer but rather knowledge of what to do in order to produce an answer. Operative knowledge is constructive and, consequently, is best demonstrated in situations where something new is generated, something that was not already available to the operator. The novelty that matters is, of course, novelty from the subject's point of view. An observer, experimenter, or teacher can infer this subjective novelty, not from the correctness of a response but from the struggle that led to it. It is not the particular response that matters but the way in which it was arrived at.

In the preceding pages, I have several times used the term "interpretation". I have done it deliberately, because it focuses attention on an activity that requires awareness and deliberate choice. Although all the material that is used in the process of interpreting may have been shaped and prepared by prior interaction with experiential things and with people, and although the validation of any particular interpretation does, as we have seen, require further interaction, the process of interpreting itself requires reflection. If an organism does no more than act and react, it would be misusing the word to say that the organism is interpreting. Interpretation implies awareness of more than one possibility, deliberation, and rationally controlled choice.

A student's ability to carry out certain activities is never more than part of what we call "competence". The other part is the ability to monitor the activities. To do the right thing, is not enough; to be competent, one must also know what one is doing and why it is right. That is perhaps the most stringent reason why longitudinal observation and Piaget's clinical method are indispensable if we want to find out anything about the reflective thought of children, about their operative knowledge, and about how to teach them to make progress towards competence.

At the beginning of this talk, I mentioned that a useful analogy might be found in the teaching of athletic skills. What I was alluding to are the recently developed methods that make it possible for athletes to see what they are doing. Some of these methods involve stachistoscropy and are very sophisticated, others are as simple as the slow-motion replay of movies and videotapes. Their purpose is to give performers of intricate actions an opportunity to observe themselves act. This visual feedback is a far more powerful didactic tool than instructions that refer to details of the action which, normally, are dimly or not at all perceived by the actor himself.

The proficiency of good athletes springs to a large extent from the fact that they have, as it were, automated much of their action. As long as their way of acting is actually the most effective for the purpose, this automation is an advantage because it frees the conscious mind to focus on higher levels of control. When, however, something must be changed in the routine, this would be difficult, if not impossible to achieve without awareness of the individual steps. Hence, the efficacy of visual feedback.

Although the speed of execution that comes with automation may be a characteristic of the expert calculator, the primary goal of mathematics instruction has to be the student's conscious understanding of what he or she is doing and why it is being done. This understanding is not unlike the self-awareness the athlete must acquire in order consciously to make an improvement in his physical routine. Unfortunately we have no stachistoscope or camera that could capture the dynamics, the detailed progression of steps, of the mental operations that lead to the solution of a numerical problem. Yet, what the mathematics teacher is striving to instill into the student is ultimately the awareness of a dynamic program and its execution -- and that awareness is in principle similar to what the athlete is able to glean from a slow-motion re-presentation of his or her own performance. In the absence of any such technology to create self-reflection, the teacher must find other means to foster operative awareness. At the present state of the art, the method of the "teaching experiment" developed by Steffe seems to be the most hopeful step in that direction (Steffe, 1977; Steffe & Cobb, 1983).

The term "teaching experiment" could easily be misunderstood. It is not intended to indicate an experiment in teaching an accepted way of operating, as for instance, the adult's way of adding and subtracting. Instead, it is primary an exploratory tool, derived from Piaget's clinical interview and aimed at discovering what goes on in the student's head. To this it adds experimentation with ways and

means of modifying the student's operating. The ways and means of bringing about such change are, in a sense, the opposite of what has become known as behavior modification.

A large part of educational research has been employing a procedure that consists of setting tasks, recording solutions, and analysing these solutions as though they resulted from the child's fumbling efforts to carry out operations that constitute an adult's competence. The teaching experiment, instead, starts from the premise that the child cannot conceive of the task, the way to solve it, and the solution in terms other than those that are available at the particular point in the child's conceptual development. The child, to put it another way, must interpret the task and try to construct a solution by using material she already has. That material cannot be anything but the conceptual building blocks and operations that the particular child has assembled in her own prior experience.

Children, we must never forget, are not repositories for adult "knowledge" but organisms which, like all of us, are constantly trying to make sense of, to understand their experience.

Most traditional measurements of student learning in mathematics assume that individual differences in students' concepts either vary insubstantially or are unimportant in their influences on the mathematics studied. ... In contrast, if one assumes that there are a variety of ways of understanding a concept mathematically, individual differences in mathematics become the diversity in students' understandings of concepts or of mathematics itself. The clinical interview provides a means for searching for and exploring these individual understandings.

(Confrey, 1980)

It is not in the least facetious to say that the interview goal is to gain understanding of the child's understandings. The difference between the child interpreting (and trying to solve) a task in the given context, and the interviewer interpreting the child's responses

and behavior in the context of the task, is that the interviewer can test his interpretation by deliberately modifying certain elements in the child's experiential field. The interviewer can also ask questions and see whether or not the responses are compatible with his or her conjectures about the child's conception of what is going on. Whenever an incompatibility crops up, the interviewer's conjectures have to be changed and their replacements tested again, until at last they remain viable in whatever situations the interviewer can think of and create.

In short, the interviewer is constructing a model of the child's notions and operations. Inevitably, that model will be constructed, not out of the child's conceptual elements, but out of conceptual elements that are the interviewer's own. It is in this context that the epistemological principle of fit, rather than match, is of crucial importance. Just as cognitive organisms can never compare their conceptual organizations of experience with the structure of an independent objective reality, so the interviewer, experimenter, or teacher can never compare the model he or she has constructed of a child's conceptualizations with what actually goes on in the child's head. In the one case as in the other, the best that can be achieved is a model that remains viable within the range of available experience.

The teaching experiment, as I suggested before, is, however, something more than a clinical interview. Whereas the interview aims at establishing "where the child is", the experiment aims at ways and means of "getting the child on". Having generated a viable model of the child's present concepts and operations, the experimenter hypothesizes pathways to guide the child's conceptualizations towards adult competence. In order to formulate any such hypothetical path, let alone implement it, the experimenter/teacher must not only have a model of the student's present conceptual structures but also an analytical model of the adult conceptualizations towards which his guidance is to lead.

The structure of mathematical concepts is still largely obscure.⁽⁵⁾ This may seem a strange complaint, given the amount of work that has been done in the last hundred years to clarify and articulate the foundations of mathematics. As a result of that work there is no shortage of definitions, but these definitions, for the most part, are formal rather than conceptual. That is, they merely substitute other signs or symbols for the definiendum. Rarely, if ever, is there a hint, let alone an indication, of what one must do in order to build up the conceptual structures that are to be associated with the symbols. Yet, that is of course what a child has to find out if it is to acquire a new concept.

Let me give you an example. A current definition of number, in the sense of "positive integer" says that it is "a symbol associated with a set and with all other sets which can be put into one-to-one correspondence with this set." (James & James, 1968). The mention of "put" makes it sound like an instruction to act, a directive for construction, which is what it ought to be. But, in order to begin that construction, the student would have to have a clear understanding of "set" and, more important still, of "one". Such understanding can be achieved only by reflecting on the operations of one's own mind and the realization that with these operations one can create units and sets anywhere and at any time, irrespective of any sensory signals. That means that, rather than speak of "sets" and "mathematical objects" as though they had an independent existence in some "objective" reality, teachers would have to foster the student's reflective awareness of his or her mental operations, because it is only from these that the required concepts can be abstracted.

The teaching experiment, at any rate, presupposes an explicit model of adult functioning. The experimental part of the method then consists in a form of "indirect guidance" aimed at modifying the child's present concepts and operations (which the experimenter "knows" in terms of the model constructed on the basis of observing

the particular child) towards the adult concepts and operations (which the experimenter "knows" in terms of the model constructed on the basis of analysing the adult procedures). Because the child necessarily interprets verbal instructions in terms of her own experience, the "guidance" must take the form either of questions or of changes in the experiential field that lead the child into situations where her present way of operating runs into obstacles and contradiction. Analogous to the adult who organizes general experience, the child is unlikely to modify a conceptual structure unless there is an experience of failure or, at least, surprise at something not working out in the expected fashion. Such failure or surprise, however, can be experienced only if there was an expectation -- and that brings me to the last point I want to make.

If I have had any success at all presenting the constructivist epistemology as a possible basis for education and educational research, this last point will be easy to make and its importance should become obvious.

The more abstract the concepts and operations that are to be constituted, the more reflective activity will be needed. Reflection, however, does not happen without effort. The concepts and operations involved in mathematics are not merely abstractions, but most of them are the product of several levels of abstraction. Hence, it is not just one act of reflection that is needed, but a succession of reflective efforts -- and any succession of efforts requires solid motivation.

The need for motivation is certainly no news to anyone who has been teaching. How to foster motivation has been discussed for a long time. But here again, I believe, the effect of behaviorism has been profoundly detrimental. The basic dogma of behaviorism merely says that behavior is determined by the consequences it has produced in the past (which is just another way of saying that organisms operate

inductively). There is every reason to agree with that. The trouble arises from the usual interpretation of "reinforcement", i.e. of the consequence that is rewarding and thus strengthens specific behaviors and increases the probability of their recurrence.

There is the wide-spread misconception that reinforcement is the effect of certain well-known commodities such as cookies, money, and social approval. It is a misconception, not because organisms will not work quite hard to obtain these commodities, but because it obscures the one thing that is often by far the most reinforcing for a cognitive organism: to achieve a satisfactory organization, a viable way of dealing with some sector of experience. This fact adds a different dimension to the conception of reinforcement because whatever constitutes the rewarding consequence in these cases is generated wholly within the organism's own system.

Self-generated reinforcement has an enormous potential in cognitive, reflective organisms. (All of us, I am sure, have spent precious time and sweat on puzzles whose solution brought neither cookies, nor money, and negligible social approval.) But this potential has to be developed and realized.

When children begin to play with wooden blocks, they sooner or later place one upon another. Whatever satisfaction they derive from the resulting structure, it provides sufficient incentive for them to repeat the act and to improve on it. They may, for instance, implicitly or explicitly set themselves the goal of building a tower that comprises all the blocks. If they succeed, they are manifestly satisfied, irrespective of tangible rewards or an adult's comment, for they build towers also in the absence of observers. The reward springs from the achievement, from the successful deliberate imposition of an order that is inherent in their own way of organizing. To repeat the feat, the tower has to be knocked down. That, too,

turns out to be a source of satisfaction because it once more provides evidence of the experiencer's power over the structure of experience.

To some, these observations may seem trivial. To me, they exemplify a basic feature of the model of the cognitive organism, a feature that must be taken into account if we want to educate.

From the constructivist point of view, it makes no sense to assume that any powerful cognitive satisfaction springs from simply being told that one has done something right, as long as "rightness" is assessed by someone else. To become a source of real satisfaction, "rightness" must be seen as the fit with an order one has established oneself. Teachers as well as mathematicians tend to assume that there exists in every particular case an objective problem and an objectively "true" solution. Children and students of any age are therefore expected somehow to come to "see" the problem, its solution, and the necessity that links the two. But the necessity is conceptual and it can spring from nothing but the awareness of the structures and operations involved in the thinking subject's conceptualization of the problem and its solution. Logical or mathematical necessity does not reside in any independent world -- to see it and gain satisfaction from it, one must reflect on one's own constructs and the way in which one has put them together.

Final Remarks

Educators share the goal of generating knowledge in their students. However, from the epistemological perspective I have outlined, it appears that knowledge is not a transferable commodity and communication not a conveyance.

If, then, we come to see knowledge and competence as products of the individual's conceptual organization of the individual's experience,

the teacher's role will no longer be to dispense "truth" but rather to help and guide the student in the conceptual organization of certain areas of experience. Two things are required for the teacher to do this: on the one hand, an adequate idea of where the student is and, on the other, an adequate idea of the destination. Neither is accessible to direct observation. What the student says and does can be interpreted in terms of a hypothetical model -- and this is one area of educational research that every good teacher since Socrates has done intuitively. Today, we are a good deal closer to providing the teacher with a set of relatively reliable diagnostic tools.

As for the helping and guiding, good teachers have always found ways and means of doing it because, consciously or unconsciously they realized that while one can point the way with words and symbols, it is the student who has to do the conceptualizing and the operating.

That leaves the destination, the way of operating that would be considered "right" from the expert's point of view. As I have mentioned earlier, a conceptual model of the formation of the structures and the operations that constitute mathematical competence is essential because it, alone, could indicate the direction in which the student is to be guided. The kind of analysis, however, that would yield a step by step path for the construction of mathematical concepts has barely been begun. It is in this area that, in my view, research could make advances that would immediately benefit educational practice. If the goal of the teacher's guidance is to generate understanding, rather than train specific performance, his task will clearly be greatly facilitated if that goal can be represented by an explicit model of the concepts and operations that we assume to be the operative source of mathematical competence. More important still, if students are to taste something of the mathematician's satisfaction in doing mathematics, they cannot be expected to find

it in whatever rewards they might be given for their performance but only through becoming aware of the neatness of fit they have achieved in their own conceptual construction.

Notes

- (1) For one, Parmenides, the real world was an indivisible, static whole, for others, Leucippus and Democritus, the real world was a mass of constantly moving atoms.
- (2) An excellent exposition can be found in Popkin, R.H., The history of scepticism from Erasmus to Spinoza. Berkeley, CA: University of California Press, 1979.
- (3) Berkeley has been interpreted in this way by Popper (*op. cit.* chapters 3 & 6). Vico's instrumentalism is explicit in his De antiquissima Italorum sapientia of 1710 (Naples: Stamperia de' Classici, 1858). Hume's analysis of induction is instrumentalist and Kant's program of "transcendental inquiry" into the workings of reason in his Critique supplied powerful ammunition to the modern instrumentalists; cf. Mach. E., The Guiding principles of my scientific theory of knowledge (1910), in S. Toulmin, Physical reality (New York: Harper Torchbooks, 1970) and Simmel, G., Haupt-probleme der Philosophie (Berlin: de Gruyter, 7th edition, 1950).
- (4) If this isolating of the named thing or "referent" is a demanding task with relatively simple perceptual compounds, such as apple, it is obviously much more difficult when the meaning of the word is a concept that requires further abstraction from sensory experience or from mental operations. But since we want to maintain that words such as "all" and "some", "mine" and "yours", "cause" and "effect", "space" and "time", and scores of others have meaning, we must assume that these meanings, though they cannot be directly perceived, are nevertheless somehow isolated and made retrievable by every learner of the language.
- (5) For recent conceptual analyses see Steffe, L.P., von Glasersfeld, E., Richards, J., & Cobb, P. Children's counting types: Philosophy, theory, and application. New York: Praeger, 1983.

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L'APPRENTISSAGE EN TANT QU'ACTIVITE CONSTRUCTIVE

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Le thème général que l'on m'a demandé de traiter est "La recherche en didactique de la mathématique dans une perspective épistémologique". Ceci ne semble pas plus menaçant que tant d'autres thèmes académiques. Mais ne vous laissez pas tromper par l'apparence anodine du discours. Introduire des considérations épistémologiques dans une discussion portant sur l'éducation a toujours été de la dynamite. Socrate l'a fait, et on lui fit rapidement boire de la ciguë. Giambattista Vico le fit aussi au XVIII^e siècle, et l'establishment philosophique s'empressa de s'en débarrasser. A notre propre époque il y eut Jean Piaget. Son intention était de réellement demeurer à l'écart de l'éducation mais il s'y laissa attirer — et nous savons quel fut le sort de son épistémologie aux mains des interprètes et des traducteurs. Il semble que discuter d'éducation d'un point de vue épistémologique était un moyen assuré de se suicider au plan intellectuel. Toutefois, ces derniers temps, le monde de l'éducation a peut-être commencé à changer. Du moins, la discipline particulière représentée à la présente rencontre, la discipline s'intéressant aux nombres, à l'arithmétique, et éventuellement à la mathématique, manifeste les symptômes d'une volonté de changer.

Les changements rapides qui se sont produits depuis quelques décennies dans les méthodes d'enseignement de la mathématique — d'un associationnisme simpliste aux "Nouvelles Mathématiques" et au "Retour aux habiletés de base" — ne produisirent pas les miracles auxquels l'on s'attendait. Leur échec créa un climat ne déclenchant guère d'enthousiasme pour de nouvelles modes. Aujourd'hui, je crois qu'il est juste de dire qu'il y a un désillusionnement plus ou moins général. Ce désillusionnement est sain et propice puisqu'il nous amène plus près du point où nous pourrions être prêts à repenser quelques-unes des présuppositions fondamentales inhérentes aux théories traditionnelles en éducation. Parmi ces présuppositions se trouvent nos conceptions de l'enseignement et de l'apprentissage, ainsi que celle, la plus fondamentale de toutes, la conception de ce que c'est que "connaître".

Il y a dix ou quinze ans il aurait été presque inconcevable de soumettre des éducateurs ou des chercheurs en éducation à un exposé se proposant de traiter d'une théorie de la connaissance. La préoccupation des éducateurs était de faire pénétrer les connaissances dans la tête de leurs étudiants, et celle des chercheurs en éducation, de trouver de meilleures façons d'y parvenir. Il n'y avait alors que peu d'incertitude, si toutefois il en existait, en ce qui avait trait aux connaissances que les étudiants devaient acquérir, et l'on était convaincu que d'une façon ou d'une autre, la connaissance pouvait être transmise d'un enseignant à un étudiant. La seule question était de savoir quelle était le meilleur moyen de réaliser ce transfert - et les chercheurs en éducation, avec leurs tests critériés et leurs méthodes statistiques sophistiquées, devaient apporter la réponse définitive.

Apparemment quelque chose n'a pas fonctionné. Les choses ne se passèrent pas telles que prévues. L'on est maintenant désappointé, et ce désappointement - et je tiens à insister sur ceci - n'est pas restreint à la didactique de la mathématique, mais touche aussi l'enseignement et les méthodes didactiques de pratiquement toutes les matières. A ma connaissance, il n'y a qu'une seule exception, laquelle offre un contraste remarquable: l'enseignement des habiletés physiques, et spécialement des habiletés athlétiques. Dans ce domaine il n'y a aucune cause de désappointement. Durant ces mêmes dix ou quinze ans, pendant lesquels l'enseignement de matières intellectuelles a en quelque sorte échoué, l'enseignement d'habiletés telles que le tennis, le ski, le saut à la perche, et le lancer du javalot, a littéralement avancé par sauts et par bonds. Le contraste n'est pas seulement spectaculaire, mais il est aussi révélateur. Je reviendrai sur ce phénomène un peu plus tard lorsque, je l'espère, nous pourrons présenter une analogie qui, pour le moment, pourrait paraître trop absurde.

Si les efforts en éducation s'avèrent réellement défailants, les présuppositions sur lesquelles, implicitement ou explicitement, ces

efforts étaient fondés, doivent être remises en question et il semble tout à fait raisonnable de suggérer, à l'instar de ceux qui ont choisi ce sujet comme thème de discussion, que nous commençons par examiner cette matière que l'éducation prend comme champ d'activités, c'est-à-dire "la connaissance".

La présente communication a trois objectifs. Premièrement, je reviendrai à ce que je considère être l'origine des difficultés que nous avons eues avec la conception traditionnelle de la connaissance. Cette revue historique sera non seulement une ébauche, mais elle sera également biaisée car j'entretiens des idées bien arrêtées sur le sujet. Cependant, considérant l'état lamentable dans lequel la théorie de la connaissance s'est retrouvée durant les dernières cinquante années dans les sciences "exactes", mon effort, je l'espère, ne sera pas jugé injustifié.

Deuxièmement, je proposerai une conceptualisation de la "connaissance" qui évite ces problèmes, et qui, de plus, apporte un autre bénéfice en ce sens qu'elle jette une lumière salutaire sur le processus de communication. Comme je l'ai dit il y a quelques instants, à titre d'enseignants nous désirons engendrer des connaissances chez les étudiants. Après tout, c'est pourquoi nous sommes rémunérés et puisque l'acquisition guidée de connaissances, peu importe la façon de l'envisager, semble reposer sur un processus de communication, nous devrions nous intéresser à la façon dont ce processus pourrait fonctionner. Selon mon expérience, ceci est un aspect auquel peu de réflexion a été apportée. Les professeurs en éducation ont consacré, et consacrent avec raison beaucoup de temps et d'efforts, au curriculum. C'est-à-dire, qu'ils font de leur mieux pour déterminer ce qu'il faut enseigner, ainsi que la séquence dans laquelle cela devrait être fait. Toutefois le processus sous-jacent de la communication linguistique, le processus duquel dépend leur enseignement, est, la plupart du temps, simplement pris pour acquis. On a eu une confiance naïve dans le langage et dans son efficacité. Même s'il ne faut pas beaucoup de temps à un bon professeur pour

découvrir qu'il ne suffit pas de dire les choses pour "les faire comprendre", il y a très peu, s'il en existe, de vision théorique sur la raison pour laquelle la communication linguistique n'accomplit pas tout ce qu'elle est supposée faire. La théorie de la connaissance que je propose, même si elle ne résout certainement pas tous les problèmes, clarifie très bien ce problème particulier.

Finalement, après avoir proposé ce que j'aimerais appeler un modèle de "connaître" qui incorpore une vision spécifique du processus de transmission de la connaissance, j'explorerai brièvement une façon d'appliquer ce modèle à la chose à laquelle, nous tous ici, sommes intéressés: comment initier les enfants à l'art, au mystère, et à la satisfaction merveilleuse que procurent les opérations numériques.

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Déjà au VI^e siècle avant Jésus-Christ la nature de la connaissance était un problème chaudement débattu. Le débat s'est poursuivi de façon plus ou moins continue, et même si sous divers aspects, il fut haut de couleur à un certain point de vue, il s'est avéré remarquablement monotone. Durant tout ce temps le problème central est demeuré sans solution et les arguments, qui au début créèrent la difficulté majeure, sont ceux-là mêmes qui encore aujourd'hui empêchent tout règlement de la question.

L'histoire débute avec les premiers documents sur l'épistémologie à nous être parvenus, les soi-disant "fragments" de l'ère pré-socratique. Les idées avec lesquelles ces hommes se sont débattus, et qu'ils ont tenté de clarifier, sont certainement apparues quelque temps avant leur époque, mais comme nous ne possédons aucun document antérieur écrit, cette période est très nébuleuse. Quoi qu'il en soit, il est très peu probable que le degré de sophistication exhibé par les pré-socratiques ait pu être atteint en une ou deux générations. Quelque fragmentaires qu'ils soient, leurs discours ne laissent aucun doute sur le fait que vers la fin du V^e siècle avant Jésus-Christ, le processus impliqué dans "connaître" a été conceptuellement encadré en un scénario général relativement stable. En

général, les penseurs intéressés à l'activité de connaître acceptaient tacitement le scénario selon lequel le connaisseur et les choses qu'il vient à connaître, sont, dès le départ, des entités séparées et indépendantes.

J'insiste sur le fait que cette dichotomie ne coïncide pas avec la distinction entre le sujet connaissant et les connaissances du sujet. Cette deuxième dichotomie apparaît chaque fois qu'un acteur devient conscient de son activité propre, ou lorsqu'un penseur commence à réfléchir sur sa propre pensée. Ce second problème relatif à la prise de conscience de sa propre pensée n'est pas identique au problème de cognition. Même si les deux sont reliés, dans le sens qu'ils se croisent (e.g., dans une analyse de la pensée réflexive qui interviendra dans la discussion un peu plus loin), je veux traiter ici la première seulement. Quoi qu'il en soit, les pré-socratiques prenaient pour acquis l'habileté des humains d'être conscients de connaître. La question qui commençait à les préoccuper était de savoir comment il pouvait être possible d'en venir à connaître le monde. C'est dans la poursuite d'une telle question que le scénario cognitif qu'ils acceptèrent, et qui a été perpétué par à peu près tous les épistémologues qui suivirent, se révèle d'une importance si décisive. Une fois choisi comme base pour l'édification d'une théorie de la connaissance, ce scénario a engendré un paradoxe. Ce paradoxe est inévitable et depuis les 2500 ans écoulés, il a hanté sans cesse les philosophes.

La raison pour laquelle ce scénario cognitif en particulier fut adopté est très simple. Il est le reflet de la situation telle qu'elle apparaît initialement à notre expérience. La question "Comment pouvons-nous connaître quelque chose" a peu de chance d'être posée au début du développement de celui qui apprend. Serait bien bizarre l'enfant de six ans qui, en revenant à bicyclette de l'école se poserait soudainement la question: "Par quel sortilège est-ce que je réussis à trouver le chemin de la maison?" ou "Qu'est-ce qui s'est passé exactement lorsque j'ai appris à conduire ma bicyclette?"

Je n'insinue pas ici que ce sont là des questions qu'un enfant de six ans, ou en fait n'importe qui devrait poser. Je dis seulement que si jamais nous les posons, ce sera à un âge plus avancé. Il en est de même pour les questions "Comment se fait-il que je sais ce que je connais?". Ceux qui ont éprouvé une telle curiosité épistémologique ont probablement formulé leur première question pertinente aux alentours de leur quinzième année ou plus tard. C'est-à-dire qu'ils ont commencé à se questionner sur leurs connaissances à un moment de leur carrière cognitive où ils avaient déjà acquis un bagage énorme de savoir-faire et de connaissances. Inévitablement, ils supposaient tacitement que presque tout ce qu'ils savaient à ce moment était des connaissances reliées à l'environnement, au monde dans lequel ils vivaient. Il n'y a rien de surprenant à ce qu'il en fut ainsi. Une fois que l'on a appris à se débrouiller avec certaines choses, il n'y a aucune raison de soupçonner que celles-ci puissent ne pas être ce qu'elles semblent être.

Si une personne dont la connaissance n'a cessé de croître et de s'étendre à travers les ans, en vient à s'interroger sur la façon dont on acquiert toutes ces connaissances, il semble raisonnable de postuler qu'au début, à l'instar d'un explorateur qui viendrait dans une terra incognita, le sujet à l'état inexpérimenté et complètement ignorant, vient dans ce monde, avec à la fois le besoin et la volonté de découvrir ce qu'est ce monde. Les premiers, sinon les seuls, des outils qui semblent disponibles pour une telle tâche, sont évidemment les sens. Il s'ensuit que les sens sont immédiatement catalogués d'organes, ou canaux, par lesquels celui qui vit des expériences reçoit des messages de l'environnement. En se basant sur ces messages, celui-ci doit donc, et apparemment il le peut, se construire une "image" du monde. Dans notre jargon contemporain ceci est souvent exprimé en disant que les sens transmettent l'information au sujet qui vit des "expériences" de se faire une représentation du monde. Habituellement ceci semble très bien fonctionner. Bien entendu, à l'occasion, les sens s'avèrent quelque peu

décevants, mais en gros ils fonctionnent assez bien pour que nous puissions nous aménager un modus vivendi. A la condition de demeurer patients et flexibles, nous continuerons à faire des ajustements, et aussi longtemps que les choses fonctionnent modérément bien, il n'y aura nul besoin de mettre en doute la validité globale de l'image qu'on a réussi à élaborer.

Les pré-socratiques ont débuté de cette façon tout à fait normale, mais comme il se trouvait parmi eux des penseurs très originaux, ils se retrouvèrent avec des images du monde mutuellement incompatibles.⁽¹⁾ Evidemment ceci fut perçu comme un problème et souleva deux questions intimement reliées: la première, comment quelqu'un pourrait-il, à partir des messages sensoriels, se composer une image du monde, et la deuxième, comment quelqu'un pourrait-il être certain qu'une image particulière du monde soit "vraie"? Les tentatives pour répondre à ces questions se heurtèrent très tôt à des difficultés, quelques-unes d'entre elles si sérieuses qu'elles n'ont pas encore été surmontées.

Je tiens ici à m'attacher au second problème parce qu'il est inhérent et inévitable dans le scénario de celui qui découvre. Si l'expérience est le seul contact que l'être connaissant puisse avoir avec le monde, il n'y a aucun moyen de comparer les produits de l'expérience avec la réalité, source supposée de tous les messages que nous recevons. La question à savoir, à quel point véridique peut être la connaissance acquise, ne peut donc pas être tranchée. Pour y répondre, il faudrait comparer ce que l'on connaît à ce qui existe dans le monde "réel" -- et pour ce faire il faudrait savoir ce qui "existe". Alors le paradoxe est le suivant: pour évaluer si votre connaissance est vraie, il vous faudrait connaître ce que vous parvenez à connaître avant de parvenir à le connaître.

L'argument, à l'effet que ce n'est qu'en regardant à la fois une image et ce qu'elle est supposée représenter que l'on peut évaluer la ressemblance ou la fiabilité de cette image, avait déjà été

apporté au temps des pré-socratiques et il a été depuis l'assise principale de tout scepticisme. L'histoire de l'épistémologie occidentale est l'histoire des essais plus ou moins imaginatifs pour le contourner. Aucune de ces tentatives ne s'est avérée satisfaisante. Le génie poétique de Platon réussit presque à éliminer ce dilemme en diminuant le rôle de l'expérience. Il relégua la vraie réalité dans le monde des idées et fit de l'expérience sensorielle une affaire secondaire, trouble, non fiable, et finalement non pertinente dans la recherche de la vérité. Puisque le monde des idées n'était accessible qu'à l'esprit réfléchissant, cette conception engendra la notion de solipsisme, la notion qu'il n'y a pas de monde réel au-delà de celui que l'esprit se crée pour lui-même.

Comme alternative l'on pourrait, comme Descartes le suggéra, placer sa foi en Dieu et croire que le divin créateur n'a pas été assez malicieux pour avoir affublé ses créatures de sens trompeurs.

Aucune de ces deux alternatives ne fournit de solution durable. Le solipsisme devient une absurdité chaque fois qu'une idée que nous avons conçue est anéantie par l'expérience. En fait, ceci n'est pas rare. Nous sommes constamment confrontés au fait que le monde dans lequel nous vivons n'est pas tout à fait le monde que nous désirons et qu'il existe vraiment une "réalité" brutale qui ne pardonne pas et avec laquelle nous devons composer. D'autre part, la notion cartésienne se réduit à une simple incitation à croire, et cela ne satisfait pas les besoins du philosophe. Si l'épistémologie doit être fondée sur la foi aveugle en nos sens, lesquels procuraient une image vraie, elle ne peut accomplir ce qu'elle se propose de faire, c'est-à-dire fournir une base rationnelle pour la génération et l'évaluation de la connaissance. De fait, l'incitation de Descartes à avoir confiance en nos sens, lesquels viennent de Dieu, ne fait que déplacer le problème. Si l'on pense que les sens sont dignes de foi, le fait qu'à partir de leurs messages, nous tirons si souvent la mauvaise conclusion, devrait montrer qu'il y a une véritable difficulté d'interprétation; et si nous ne pouvons être certains

de la façon d'interpréter ce que les sens nous disent, nous devons à nouveau admettre que nous n'avons pas une connaissance certaine du monde et que l'image que nous parvenons à nous en faire demeure douteuse.

Tel que je l'ai suggéré au début, le problème est intrinsèque au scénario traditionnel. Il découle de la conception "iconique" de la connaissance, une conception exigeant une congruence ou une correspondance entre les structures cognitives et ce que ces structures doivent représenter. Selon cette conception, la vérité devient inévitablement la congruence parfaite, la représentation sans défaut. Dès que nous acceptons cette vision nous commençons à ressentir le besoin d'évaluer à quel degré nos structures cognitives coïncident avec ce qu'elles devraient représenter. Mais cette "réalité" demeure à jamais de l'autre côté de notre interface expérientielle. Pour faire une telle évaluation de la vérité nous devrions être en mesure, pour reprendre l'expression récente de Hilary Putnam, d'adopter "la perspective de Dieu" (PUTNAM, 1981). Puisque nous ne sommes pas, et logiquement nous ne pouvons pas être, dans une position permettant une telle vision du monde "réel" et de sa présumée représentation, il n'existe aucune issue à ce dilemme. Ce dont nous avons besoin est un scénario différent, une conception différente de ce qu'est "connaître", une conception selon laquelle la qualité de la connaissance n'est pas basée sur la ressemblance ou la représentation.

La première proposition explicite d'une approche différente a vu le jour dans les milieux les plus préoccupés de la foi et de sa préservation. Quand pour la première fois l'idée révolutionnaire que la terre n'était peut-être pas le centre de l'Univers a menacé sérieusement l'image du monde que l'Eglise considérait comme indiscutable et sacrée, ce furent les défenseurs de la foi qui proposèrent un scénario de rechange pour la poursuite de la connaissance scientifique. Dans sa préface du traité de Copernic, *De Revolutionibus*, Oslander (1627) suggéra:

Il n'est pas nécessaire que ces hypothèses soient vraies, ou même qu'elles ressemblent un tant soit peu à la vérité; une chose leur suffit -- c'est qu'elles fournissent plutôt des calculs qui soient conformes aux observations.

(POPPER, 1968)

(traduction libre)

Ceci introduit la notion d'une deuxième forme de connaissance, autre que la foi et le dogme, une connaissance qui soit conforme aux observations. Il s'agit de connaissances que la raison humaine tire de l'expérience. Elle ne représente pas une image du monde "réel", mais fournit à l'expérience une structure et une organisation. A ce titre, elle a une fonction des plus importantes: elle nous permet de résoudre des problèmes tirés de l'expérience.

Du temps de Descartes cette théorie instrumentaliste de la connaissance fut formulée et développée par Mersenne et Gassendi.⁽²⁾ Elle fut alors élargie par Berkeley et Vico, puis fortement appuyée, quoique non intentionnellement, par Hume et Kant; puis, à la fin du siècle dernier, elle fut appliquée à la physique et à la science en général par Ernst Mach, et à la philosophie par Georg Simmel.⁽³⁾ Cette théorie n'était pas, et n'est toujours pas, populaire auprès des philosophes traditionnels. L'idée que la connaissance est une bonne connaissance si et quand elle résout nos problèmes n'est pas un critère acceptable par ceux qui continuent d'espérer qu'éventuellement la connaissance permettra au moins l'approximation d'une image du monde "réel".

Karl Popper, qui a donné un compte rendu lucide des débuts de l'instrumentalisme, (POPPER, 1968) a lutté dur pour nous convaincre que, même si elle est raisonnable, cette théorie n'est pas satisfaisante. Comme il le redit dans son dernier livre:

Ce que nous cherchons, en science, ce sont

des théories vraies -- des énoncés vrais, des descriptions vraies de certaines propriétés structurales du monde dans lequel nous vivons. Ces théories ou systèmes d'énoncés peuvent avoir une utilité instrumentale; cependant, ce que nous cherchons en sciences n'est pas tant l'utilité que la vérité: des approximations de la vérité; une capacité à expliquer, et la capacité à résoudre des problèmes; et par conséquent, la compréhension. (POPPER, 1982)

(traduction libre)

Ceci suggère que "les descriptions", "les explications", et "la compréhension" peuvent en effet saisir des aspects du "monde dans lequel nous vivons". Que nous puissions ou non être d'accord avec cet énoncé dépendra de la façon dont nous définissons "le monde dans lequel nous vivons". Il n'y a aucun doute que Popper faisait référence à un monde objectif, c'est -à-dire un monde déjà fait dans lequel nous naissons et que, comme des explorateurs, nous devons apprendre à connaître. Voilà la vision réaliste traditionnelle, et Popper fait de son mieux pour la défendre, en dépit de tous les arguments que l'on puisse lui opposer. Une fois de plus les réalistes et les sceptiques sont dans l'impasse familière.

Cependant une autre possibilité existe. "Le monde dans lequel nous vivons" peut aussi être interprété comme le monde de nos expériences, le monde tel que nous le voyons, l'entendons et le ressentons. Ce monde ne consiste pas en "des faits objectifs" ou "des choses en soi" mais en des invariants et des constantes que nous sommes en mesure d'estimer à partir de notre expérience personnelle. Accepter une telle lecture, toutefois, revient à adopter, pour l'acte de connaître, un scénario radicalement différent. D'un explorateur condamné à chercher "des propriétés structurales" dans une réalité inaccessible, l'organisme qui vit des expériences se transforme maintenant en un constructeur de structures cognitives dans l'intention de ré-

soudre des problèmes tels que conçus ou perçus par l'organisme. Il y a cinquante ans Piaget décrit ce scénario de la plus claire des façons: "l'intelligence organise le monde en s'organisant elle-même" (Piaget, 1937). Ce qui détermine la valeur des structures conceptuelles c'est leur adéquation à l'expérience, la qualité de leur coïncidence avec l'expérience, leur viabilité comme moyens de résoudre des problèmes parmi lesquels se trouve évidemment le problème jamais réglé de l'organisation cohérente que nous nommons la compréhension.

Le monde dans lequel nous vivons, dans cette nouvelle perspective, est toujours, et nécessairement, le monde tel que nous le conceptualisons. Comme le perçut Vico il y a longtemps, les "faits" sont fabriqués par nous-mêmes et notre façon d'en faire l'expérience, plutôt que de provenir d'un monde objectif existant indépendamment. Mais ceci ne veut pas dire que nous pouvons les construire à notre gré. Ils sont des faits viables aussi longtemps qu'ils ne viennent pas en conflit avec l'expérience, aussi longtemps qu'ils demeurent fiables dans le sens qu'ils continuent à se comporter tel que nous le prévoyons.

Il est clair que cette perception de la connaissance a des conséquences sérieuses quant à notre conception de l'enseignement et de l'apprentissage. Par-dessus tout, elle déplacera l'accent mis sur la reproduction "correcte" par l'élève de ce que fait l'enseignant, vers l'organisation réussie par l'étudiant de sa propre expérience. Mais avant de m'étendre sur ce point je veux examiner la notion très répandue que la connaissance constitue un produit qui peut être simplement communiqué.

11

La façon dont nous pensons habituellement au mot "signification" est conditionnée par des siècles de langage écrit. Nous sommes portés à penser au sens des mots dans un texte plutôt qu'au sens visé par une personne émettant des sons linguistiques. Le langage écrit et les

textes imprimés ont une persistance physique. Ils reposent sur nos pupitres ou peuvent être retirés des étagères, ils peuvent être manipulés et lus. Lorsque nous comprenons ce que nous lisons, nous avons l'impression que nous avons "saisi" le sens des mots imprimés, et nous en venons à croire que ce sens résidait dans les mots et que nous l'avons extirpé comme nous le ferions pour des noix de leurs coquilles. Nous pouvons même dire qu'une signification particulière est le "contenu" d'un mot ou d'un texte. Cette notion de mots comme contenant dans lesquels l'écrivain ou l'orateur "communique" un sens à des lecteurs ou à des auditeurs s'avère extraordinairement bien ancrée et elle semble si naturelle que nous sommes réticents à la remettre en question. Cependant, ceci est une mauvaise interprétation. Pour nous en rendre compte, nous devons revenir sur nos pas et revoir comment, au début de notre carrière linguistique, le sens des mots a été acquis.

Afin d'attacher quelque sens que ce soit à un mot, un enfant doit, en premier lieu, apprendre à isoler ce mot particulier, en tant que pattern sonore récurrent, parmi la totalité des signaux sonores disponibles. Ensuite, il doit isoler quelque chose d'autre de son champ d'expériences, quelque chose qui revient de façon plus ou moins régulière en conjonction avec ce pattern sonore. Prenez un mot ordinaire et relativement non-problématique tel que "pomme". Supposons qu'un enfant soit parvenu à le reconnaître en tant qu'item récurrent dans son expérience auditive. Supposons de plus que l'enfant se doute aussi que "pomme" est la sorte de pattern sonore qui devrait être associé avec quelqu'autre objet de son champ d'expérience. Les adultes intéressés au progrès de l'enfant au plan linguistique peuvent évidemment, à l'aide d'actions et de réactions spécifiques, l'aider dans ce processus d'association, et ils considéreront que leur "enseignement" est couronné de succès lorsque l'enfant sera parvenu à isoler de son champ d'expériences quelque chose qui lui permette de réagir d'une façon qu'ils considèrent appropriée. Une fois ceci accompli, quand l'association appropriée a été formée, il y a encore un autre pas que l'enfant doit faire avant que l'on puisse dire qu'il a acquis la signification du mot "pomme". L'enfant

doit apprendre à se re-présenter la somme de ses expériences chaque fois que le mot est prononcé, même lorsqu'aucun des éléments de ce composé n'est actuellement présent dans son champ d'expériences. En d'autres mots, l'enfant doit acquérir l'habileté d'imaginer ou de visualiser, par exemple, ce qu'il a associé au mot "pomme" chaque fois qu'il entend le pattern sonore de ce mot. (4)

Cette analyse, même si elle paraît détaillée, n'est encore rien qu'un résumé rudimentaire de certains pas indispensables dans une longue procédure d'interactions. Toutefois, dans le présent contexte cela devrait suffire pour justifier la conclusion que le composé d'éléments d'expériences constituant le concept qu'un individu a associé à un mot, ne peut être rien d'autre qu'un composé d'abstractions tirées de l'expérience personnelle de cet individu. Pour chacun de nous alors, la signification du mot "pomme" est une abstraction faite individuellement à partir de toutes les expériences vécues touchant la pomme. En d'autres termes, cette signification a une origine subjective et elle réside dans la tête d'un sujet, non pas dans le mot qui, à cause d'une association, a le pouvoir de rappeler en chacun de nous notre propre représentation subjective.

Si vous souscrivez à cette idée de subjectivité inhérente des concepts et, par conséquent, de leur signification, vous vous butez immédiatement à un problème sérieux. Si les sens des mots sont vraiment notre propre construction subjective, comment pouvons-nous possiblement communiquer? Comment pourrait-on raisonnablement croire que les représentations évoquées dans l'esprit de l'auditeur sont tant soit peu semblables aux représentations que l'orateur a en tête lorsqu'il ou elle prononce les dits mots? Cette question va droit au coeur du problème de la communication. Malheureusement la conception générale de la communication a découlé de, et a été formée par, cette idée que les mots sont des contenants de sens. Si cette notion n'est pas adéquate, celle de la conception générale de la communication ne l'est pas plus.

Le problème vient de la supposition erronée que pour communiquer,

les représentations associées aux mots utilisés doivent être les mêmes pour tous les partenaires de la communication. Pour que la communication soit considérée comme satisfaisante et comme conduisant à ce que nous appelons "la compréhension", il est bien suffisant que les représentations des communicateurs soient compatibles, dans le sens qu'elles ne s'opposent pas trop manifestement au contexte de la situation ou aux attentes du locuteur.

Un exemple simple peut aider à clarifier ceci. Supposons que Jimmy entende le mot "sirène" pour la première fois. Il demande ce que ça veut dire, à quoi on lui répond qu'une sirène est une créature ayant la tête et le torse d'une femme, et la queue d'un poisson. Pour l'imaginer Jimmy n'a nul besoin d'avoir rencontré une telle créature dans une expérience véritable. Il peut s'en construire une représentation à partir d'éléments connus à condition d'être quelque peu familier avec "femme", "poisson", et les autres mots utilisés dans l'explication, et d'avoir établi des associations entre eux et des expériences qu'il a vécues. Cependant, si l'on ne dit pas à Jimmy que chez la sirène la queue de poisson remplace les jambes de la femme, il pourrait construire un composé qui soit un bipède à queue de poisson et, par conséquent, passablement différent de la créature de la mer à laquelle on faisait allusion. Jimmy pourrait alors lire des histoires de sirène et participer pour un bon bout de temps, à des conversations à leur sujet sans avoir à corriger son image. De fait, sa fausse notion du corps de la sirène ne pourrait être corrigée que s'il venait à se trouver dans une situation où l'image d'une créature avec des jambes, ainsi qu'avec une queue de poisson, venait en conflit explicite avec une image ou avec ce que disent de la sirène les gens utilisant ce langage. Ainsi, Jimmy modifierait le concept, c'est-à-dire la signification subjective du mot, seulement si un certain contexte le forçait à le faire.

Vous pouvez maintenant vous demander, comment un contexte peut-il bien forcer quelqu'un à modifier ses concepts? L'on doit répondre à cette question, non seulement dans le cadre d'une théorie de la communication, mais aussi dans celui d'une théorie de la connaissance.

La réponse que je propose est essentiellement la même dans les deux cas.

L'hypothèse de base en est une qui vous est familière. Les organismes vivent dans un monde de contraintes. Pour survivre, ils doivent être "adaptés" ou, comme je préfère le dire, ils doivent être "viables". Ceci signifie qu'ils doivent être capables de gérer leur mode de vie en tenant compte des contraintes du monde dans lequel ils vivent. Cette idée est bien connue dans le contexte de la biologie et de l'évolution. Selon moi, le principe s'applique aussi à la cognition -- avec une différence importante. Au plan de la biologie, nous sommes intéressés aux espèces, c'est-à-dire à des collections d'organismes qui, individuellement, ne peuvent modifier leur base biologique. Mais comme elles ne sont pas toutes identiques, les espèces "s'adaptent" simplement parce que tous les individus non viables sont éliminés et ne se reproduisent pas. Sur le plan cognitif, nous avons affaire à des individus et spécifiquement à leurs "connaissances" qui, heureusement, ne sont pas immuables et très rarement fatales. Dans le but de mieux éviter les affrontements avec les contraintes du monde, l'organisme capable de connaissances tente de trouver un sens à l'expérience. Il peut modifier de manière active les façons et les moyens de s'assurer une viabilité plus grande.

Que la matière à laquelle on veut trouver un sens concerne les expériences en général ou l'expérience en particulier, ce que l'on appelle la communication, "trouver un sens", est la même activité et implique les mêmes présuppositions. La procédure est la même mais la motivation, la raison pour laquelle nous voulons trouver un sens, peut être différente.

Laissez-moi commencer par les expériences ordinaires de la vie. Quelle que soit notre façon de caractériser les organismes capables de connaître, l'un de leurs traits saillants est leur capacité d'apprendre. Fondamentalement, avoir "appris" signifie avoir tiré des

conclusions de l'expérience et agi en conséquence. Evidemment agir en conséquence implique qu'il y a certaines expériences que l'on aimerait répéter, de préférence à d'autres que l'on aimerait éviter. L'expectative qu'un tel contrôle de l'expérience puisse être acquis doit reposer sur les présomptions que 1) il est possible d'identifier des régularités dans la suite des expériences et 2) dans une certaine mesure l'expérience future sera conforme à ces régularités. Comme l'a montré David Hume, ces hypothèses sont des prérequis au processus inductif et à la connaissance qui en résulte.

Pour trouver des régularités nous devons découper notre expérience en pièces détachées de façon à pouvoir dire, après certaines opérations de rappel et de comparaison, que quelques-unes d'entre elles se répètent. La segmentation et le rappel, l'évaluation des ressemblances, et les décisions quant aux choses à considérer comme différentes, relèvent tous de notre action. Néanmoins, chaque fois qu'un résultat particulier de ces activités s'avère utile (en procurant des expériences désirables ou en évitant les indésirables) nous oublions vite que nous aurions pu découper, considérer et évaluer autrement. Lorsqu'un schème a fonctionné à plusieurs reprises, nous en venons à croire, tel que Piaget l'a remarqué, qu'il n'aurait pu en être autrement et que nous avons découvert quelque chose relatif au monde réel. En fait, nous avons tout simplement une façon viable d'organiser notre expérience. "Trouver un sens" à un collection donnée d'expériences signifie donc les avoir organisées d'une façon telle qu'elle nous permet de faire des prédictions plus ou moins fiables. En effet, c'est un fait presque universel que nous interprétons l'expérience en vue de nos attentes ou dans le but de faire des prédictions au sujet d'expériences à venir.

En contraste, "trouver un sens" à un extrait du langage n'implique généralement pas la prédiction d'expériences non linguistiques futures. Toutefois, ceci nous amène à entretenir des attentes quant au reste de l'extrait que nous n'avons pas encore entendu ou lu. Ces attentes concernent des mots et des concepts, non des actions ou d'autres événements tirés de nos expériences. Bien entendu, l'extrait du langage peut avoir comme but d'exprimer une prédiction,

par exemple, "demain il va pleuvoir", mais la façon dont cette prédiction découle de l'extrait diffère de la façon dont elle aurait pu l'être, disons, de l'observation de certains nuages dans le ciel. La différence apparaît clairement lorsque l'on fait remarquer que pour donner un sens à la phrase "demain il va pleuvoir" il est tout à fait sans importance de savoir si l'on considère ou non la possibilité de pluie. Pour "comprendre" cette phrase, il suffit de produire une structure conceptuelle qui, étant donnée notre expérience passée avec les mots et la façon dont ils sont combinés, convient à la phrase entendue. Le fait que, lorsque demain venu, il ne pleuve pas, n'invalide en rien l'interprétation de cette phrase. D'autre part si la prédiction faite à partir d'une observation du ciel n'est pas confirmée par de la pluie, nous devons conclure que notre interprétation des nuages était fautive.

Malgré cette différence entre l'interprétation de l'expérience et l'interprétation du langage, les deux possèdent une caractéristique commune importante. Les deux reposent sur l'utilisation, par celui qui interprète, de matériel conceptuel qu'il possède déjà. Dans les deux cas, "trouver un sens" signifie trouver une façon de disposer les éléments conceptuels en un pattern soumis aux exigences des contraintes spécifiques. D'un côté les contraintes sont inhérentes à la façon dont nous avons découpé et organisé l'expérience; d'un autre côté, les contraintes sont inhérentes à la façon dont nous avons appris à utiliser le langage. Dans aucun de ces cas s'agit-il de répliquer un original. Si notre interprétation de l'expérience nous permet d'atteindre notre but, nous avons bien l'impression de "savoir"; et si notre interprétation d'une communication n'est aucunement contredite par les paroles ou les actions du communicateur, nous avons bien l'impression d'avoir "compris."

Le processus de compréhension, dans le contexte de la communication, est analogue au processus "d'en venir à connaître" dans le contexte de l'expérience. Dans les deux cas c'est une question de construire, à partir d'éléments disponibles, des structures conceptuelles pouvant prendre place dans des espaces libérés de contraintes. Même

si, évidemment, il s'agit ici d'une métaphore spatiale, elle nous éclaire sur le caractère essentiel de la notion de viabilité et elle fait ressortir un autre aspect qui distingue cette notion de la notion traditionnelle de "vérité": ayant construit soit une voie viable d'action, soit une solution viable à un problème tiré de l'expérience ou soit une interprétation viable d'un extrait de discours, il n'y a jamais la moindre raison de croire que cette construction est la seule possible.

III

Lorsque j'ai entamé la section sur la communication par une discussion sur le concept de signification, il a dû devenir apparent que je ne suis pas behavioriste. Depuis environ un demi-siècle les behavioristes ont fait tous les efforts pour écarter certaines notions ayant trait au "mental" (telles que la signification, la représentation et la pensée). Il appartient aux futurs historiens d'évaluer combien de dommage cette mode insensée a pu produire. En ce qui touche l'éducation, le dommage a été inouï. Puisque le behaviorisme est loin d'être défunt, des dégâts se perpétuent, et cela de plusieurs façons. Une cause commune, toutefois, est l'hypothèse que tout ce qui compte -- peut-être même tout ce qu'il y a -- ce sont les stimuli observables et les réponses observables. Cette supposition a fort bien réussi à faire disparaître la distinction entre l'entraînement et l'éducation.

Comme j'espère l'avoir montré dans la section précédente, un enfant doit apprendre bien plus que répondre "pomme" à tout exemple concrétisant l'expérience vécue de pomme. Si c'était là la seule chose qu'il puisse faire, sa compétence linguistique demeurerait équivalente à celle d'un perroquet bien entraîné. Pour l'oiseau et son entraîneur c'est une réussite remarquable que d'être parvenu si loin. Pour l'enfant de l'homme c'est un point de départ dans la voie du développement de l'auto-régulation, de la prise de conscience et du contrôle rationnel.

En tant que didacticiens de la mathématique vous le savez mieux que quiconque. Donner des réponses correctes à des questions relatives à la table de multiplication est sans doute une réussite utile, mais en soi, ce n'est pas une manifestation de connaissance mathématique. La connaissance mathématique ne peut être réduite à un répertoire de "données" accessibles, elle concerne plutôt l'habileté de calculer des résultats nouveaux. Dans la terminologie de Piaget, elle est opératoire plutôt que figurative. Elle est le produit de la réflexion... et tandis que la réflexion en soi n'est pas observable, ses produits peuvent être inférés à partir de réponses observables.

J'emploie le mot "réflexion" dans le sens utilisé originellement par Locke, c'est-à-dire, pour l'habileté de l'esprit à observer ses propres opérations. La connaissance opératoire n'est donc pas une récupération par association d'une certaine réponse, mais plutôt la connaissance de ce qu'il faut faire pour produire une réponse. La connaissance opératoire est constructive, et en conséquence, elle est mise en évidence de la façon la plus claire dans des situations où quelque chose de nouveau est produit, quelque chose qui ne faisait pas déjà partie du répertoire du sujet. L'innovation qui importe ici est, bien entendu l'innovation du point de vue du sujet. Un observateur, expérimentateur, ou enseignant peuvent faire des inférences en ce qui touche cet élément nouveau chez le sujet, non pas à partir de l'exactitude d'une réponse, mais bien à partir du combat qui l'y a conduit. Ce n'est pas particulièrement la réponse qui compte mais la façon d'y arriver.

Dans les pages qui précèdent j'ai employé à plusieurs reprises le mot "interprétation". Je l'ai fait délibérément parce qu'il attire l'attention sur une activité qui exige une conscience et un choix délibéré de la part du sujet. Même si le matériel utilisé dans le processus d'interprétation peut avoir été formé et préparé par une interaction antérieure avec des objets et des gens lors d'expériences, et même si, comme nous l'avons vu, la validation d'une interprétation quelconque exige une interaction additionnelle, le processus d'interprétation lui-même demande de la réflexion.

Si un organisme ne fait rien d'autre que d'agir et de réagir, ce serait mal utiliser le mot que de dire de lui qu'il interprète. L'interprétation implique la prise de conscience de plus d'une possibilité, la délibération, et un choix rationnel contrôlé.

L'habileté d'un étudiant à mener à bien certaines activités ne constitue jamais plus qu'une seule partie de ce que l'on appelle "la compétence". L'autre partie est l'habileté à gérer les activités. Il ne suffit pas de faire la bonne chose: pour être compétent il faut aussi savoir ce que l'on fait et pourquoi c'est la bonne chose à faire. Voilà peut-être la raison la plus sérieuse pour laquelle l'observation longitudinale et la méthode clinique de Piaget sont indispensables si nous désirons découvrir quoi que ce soit au sujet de la pensée des enfants, de leur connaissance opérationnelle, et pour apprendre comment leur enseigner à progresser vers la compétence.

Au début de cet entretien j'ai mentionné qu'une analogie utile pouvait être trouvée dans l'enseignement des habiletés athlétiques. Je faisais alors allusion aux méthodes qui, développées récemment, permettent aux athlètes de voir ce qu'ils font. Certaines de ces méthodes ont recours à la stachistoscopie et sont très sophistiquées, d'autres sont aussi simples que la reprise au ralenti de films et de vidéos. Leur but est de permettre aux exécutants d'actions complexes de s'observer eux-mêmes en action. Cette rétroaction visuelle s'avère être un outil didactique beaucoup plus puissant que des directives se référant à des détails de l'action, lesquels sont normalement perçus que de façon confuse, s'ils le sont, par l'auteur lui-même.

Le haut niveau de performance des bons athlètes peut être attribuable en grande partie au fait qu'ils ont en quelque sorte automatisé à peu près complètement toute leur action. A partir du moment où leur comportement est en fait le moyen le plus efficace pour le but visé, cette automatisation s'avère un avantage puisqu'elle libère

la raison consciente qui peut alors se centrer sur des niveaux de contrôle plus élevés. Toutefois, lorsque quelque chose dans la routine doit être modifié, il serait difficile, voire même impossible d'y parvenir sans avoir conscience de chacune des étapes. D'où l'efficacité de la rétroaction visuelle.

Même si la rapidité d'exécution résultant de l'automatisation peut être une caractéristique des calculateurs experts, l'objectif principal de l'enseignement de la mathématique se doit d'être la compréhension consciente, par l'étudiant, de ce qu'il fait et de la raison pourquoi il le fait. Cette compréhension n'est pas différente de l'auto-conscience que l'athlète doit acquérir s'il veut améliorer sa routine sur le plan physique. Malheureusement nous n'avons ni stachistoscope, ni caméra qui puisse saisir la dynamique, le déroulement détaillé des étapes, des opérations mentales conduisant à la solution d'un problème numérique. Néanmoins, ce que le professeur de mathématiques s'efforce d'instaurer chez l'étudiant, c'est en fin de compte une prise de conscience d'un programme dynamique et de son exécution -- et cette prise de conscience est en principe semblable à ce que l'athlète glane de la re-présentation au ralenti de sa propre performance. En l'absence d'une telle technologie pour faire naître la réflexion sur soi, l'enseignant doit trouver d'autres moyens d'engendrer une prise de conscience opératoire de soi. Dans l'état présent de nos connaissances, la méthode de "l'expérimentation didactique", développée par Steffe, semble être la tentative la plus prometteuse dans cette direction. (Steffe, 1977; Cobb et Steffe, 1983).

L'expression "expérimentation didactique" pourrait facilement prêter à confusion. Il ne s'agit pas d'indiquer par là une expérience portant sur une façon acceptée d'opérer, comme par exemple, la manière d'additionner ou de soustraire des adultes. Plutôt, c'est principalement un outil d'exploration élaboré à partir de l'entrevue clinique de Piaget, dans le but de découvrir ce qui se passe dans la tête de l'élève. Ce qu'il y a d'ajouté à l'entrevue clinique,

c'est l'expérimentation avec des voies et des moyens de modifier la façon de faire de l'étudiant. Les voies et les moyens de produire un tel changement sont, dans un sens, à l'opposé de ce qui s'est fait connaître sous l'appellation "modification de comportement".

Une grande partie de la recherche en éducation a fait appel à une procédure qui consiste à définir des tâches, à enregistrer des solutions et à analyser ces solutions comme si elles résultaient des tâtonnements de l'enfant pour exécuter des opérations qui constituent une compétence d'adulte. L'expérimentation didactique, par contre, part de la prémisse que l'enfant ne peut concevoir la tâche, la façon de la résoudre, et la solution en termes autres que ceux qui lui sont accessibles en vertu de son développement conceptuel. En d'autres mots, l'enfant doit interpréter la tâche et essayer d'élaborer une solution en utilisant le matériel qu'il possède déjà. Ce matériel ne peut être autre chose que les blocs de construction conceptuels et les opérations que cet enfant a rassemblés à travers ses expériences antérieures.

Nous ne devons jamais oublier que les enfants ne sont pas les dépositaires des "connaissances" adultes mais bien des êtres qui, comme nous tous, essaient constamment de donner un sens, de comprendre leurs expériences.

La plupart des mesures traditionnelles de l'apprentissage de la mathématique chez les élèves supposent que les différences individuelles, en ce qui a trait aux concepts des étudiants, varient de façon non substantielle ou ne sont pas importantes quant à leurs influences sur les mathématiques étudiées... Par contre, si l'on suppose qu'il existe toute une gamme de façons de comprendre mathématiquement un concept, les différences individuelles en mathématique deviennent chez les étudiants,

la diversité de compréhensions de concepts ou de la mathématique elle-même. L'entrevue clinique procure un moyen de chercher et d'explorer ces compréhensions individuelles. (CONFREY, 1980) (traduction libre)

Ce n'est certainement pas une boutade de dire que le but de l'entrevue est la compréhension de la compréhension de l'enfant. La différence entre l'enfant qui interprète (et essaie de résoudre) une tâche dans le contexte donné, et l'intervieweur qui interprète les réponses et le comportement de l'enfant dans le contexte de la tâche, c'est que l'intervieweur peut tester son interprétation en modifiant délibérément certains éléments du champ expérientiel de l'enfant. L'intervieweur peut aussi poser des questions et voir si oui ou non les réponses sont compatibles avec ses conjectures relativement à la conception qu'a l'enfant de ce qui se passe. Chaque fois qu'une incompatibilité surgit, les conjectures de l'intervieweur doivent être changées et les nouvelles testées à nouveau, jusqu'à ce qu'à la fin elles demeurent viables dans toutes les situations auxquelles l'intervieweur peut penser et créer.

En un mot, l'intervieweur construit un modèle des notions et opérations de l'enfant. Inévitablement, ce modèle sera construit, non pas à partir des éléments conceptuels de l'enfant, mais bien à partir d'éléments conceptuels qui sont ceux de l'intervieweur. C'est dans ce contexte que le principe épistémologique d'ajustement, plutôt que de correspondance, joue un rôle crucial. Tout comme les êtres capables de connaissances ne peuvent jamais comparer les organisations conceptuelles qu'ils font de leurs expériences, avec la structure d'une réalité objective indépendante, de même l'intervieweur, l'expérimentateur, ou l'enseignant ne peuvent jamais comparer le modèle qu'ils ont construit des conceptualisations d'un enfant avec ce qui se passe réellement dans sa tête. Dans un cas comme dans l'autre, le mieux qui puisse être réalisé est un modèle qui demeure viable dans le champ des expériences accessibles.

L'expérimentation didactique, tel que je l'ai suggéré auparavant, possède cependant quelque chose de plus qu'une entrevue clinique. Tandis que l'entrevue vise à établir "où est situé l'enfant", l'expérimentation vise les voies et les moyens de "faire progresser l'enfant". Ayant généré un modèle viable des concepts et opérations actuels de l'enfant, l'expérimentateur pose des hypothèses quant aux chemins à utiliser pour conduire l'enfant de ses conceptualisations à lui vers une compétence d'adulte. Pour décrire l'un de ces chemins hypothétiques, sans parler de sa mise en opération, l'expérimentateur/enseignant est obligé de posséder, non seulement un modèle des structures conceptuelles présentes de l'élève, mais aussi un modèle analytique des conceptualisations de l'adulte vers lesquelles sa direction doit mener.

La structure des concepts mathématiques est encore passablement obscure.⁽⁵⁾ Considérant tout le travail qui a été fait dans les derniers cent ans pour clarifier et articuler les fondements de la mathématique, cette dénonciation peut paraître étrange. Comme résultat de ce travail il ne manque pas de définitions, mais ces définitions sont pour la plupart formelles plutôt que conceptuelles. C'est-à-dire qu'elles substituent simplement aux définitions d'autres signes ou symboles. Rarement, si jamais, y a-t-il une allusion, encore moins une indication, de ce que l'on doit faire pour construire les structures conceptuelles à être associées aux symboles. Et pourtant c'est, bien entendu, ce qu'un enfant doit découvrir s'il est pour acquérir un concept nouveau.

Laissez-moi vous donner un exemple. Une définition courante du nombre, dans le sens d'entier positif, dit que c'est "un symbole associé à un ensemble et à tous les autres ensembles qui peuvent être mis en correspondance un-un avec cet ensemble". (James et James, 1968). La mention de "mis" donne l'impression d'une instruction d'agir, une directive pour la construction, l'étudiant devrait avoir une compréhension claire de "ensemble" et, encore plus important, de "un". Une telle compréhension ne peut être atteinte que

par une réflexion sur les opérations de sa propre raison et la réalisation qu'avec ces opérations l'on peut créer des unités et des ensembles, n'importe où et en tout temps, indépendamment de tous signaux sensoriels. Ceci signifie que plutôt que de parler "d'ensembles" et "d'objets mathématiques" comme s'ils possédaient une existence indépendante dans quelque réalité "objective", les enseignants devraient encourager chez les étudiants la prise de conscience réflexive de leurs opérations mentales, parce que ce n'est qu'à partir de ces dernières que les concepts visés peuvent être abstraits.

Quoi qu'il en soit, l'expérimentation didactique présuppose un modèle explicite du fonctionnement de l'adulte. La partie expérimentale de la méthode consiste alors en une forme de "direction indirecte" visant à modifier les concepts et les opérations actuels de l'enfant (que l'expérimentateur "connaît" en terme du modèle construit à partir des observations de l'enfant concerné) pour les rapprocher des concepts et des opérations de l'adulte (que l'expérimentateur "connaît" en terme du modèle construit à partir de ses analyses des procédures de l'adulte). Parce que l'enfant interprète nécessairement les instructions verbales en terme de sa propre expérience, la "direction" doit prendre la forme soit de questions ou de changements dans le champ expérientiel qui amènent l'enfant dans des situations où sa façon d'opérer se heurte à des obstacles et à des contradictions. De façon analogue à l'adulte qui organise son expérience ordinaire, il est improbable que l'enfant modifie une structure conceptuelle, sauf s'il vit une expérience d'échec, ou du moins, de surprise face à quelque chose qui ne fonctionne pas de la façon prévue. Cependant, un tel échec ou surprise ne peuvent être vécus que s'il y a une attente — et ceci m'amène au dernier point que je veux soulever.

Si j'ai réussi, ne fut-ce qu'un tant soit peu, à présenter l'épistémologie constructive comme une base possible pour l'enseignement et la recherche en éducation, ce dernier point sera facile à présenter et son importance devrait paraître évidente.

Plus les concepts et opérations à faire construire sont abstraits, plus grande est l'activité réflexive dont on a besoin. Cependant, la réflexion ne se déclenche pas sans efforts. Les concepts et opérations en mathématique ne sont pas seulement des abstractions, mais la plupart d'entre eux sont le produit de plusieurs niveaux d'abstraction. Ce n'est pas alors un seul acte de réflexion -- qui est requis, mais une succession d'efforts de réflexion -- et toute succession d'efforts exige une motivation solide.

Le besoin de motivation n'est certainement pas quelque chose de nouveau pour quiconque a enseigné. Comment induire la motivation a été discuté depuis longtemps. Mais encore ici, je crois, l'effet du behaviorisme a été profondément préjudiciable. Selon le dogme de base du behaviorisme le comportement serait déterminé par les conséquences qu'il a produites dans le passé (qui n'est qu'une autre façon de dire que les êtres opèrent inductivement). Il y a toutes les raisons d'être en accord avec cela. La difficulté survient de l'interprétation habituelle de "renforcement", c'est-à-dire du résultat qui récompense et qui en conséquence renforce certains comportements spécifiques et accroît la probabilité de leur réapparition.

Une conception erronée très répandue veut que le renforcement résulte d'objets très connus tels que des biscuits, de l'argent, et l'approbation sociale. C'est une conception erronée, non pas parce que les êtres ne travailleront pas très fort pour obtenir ces biens, mais parce que ceci jette dans l'ombre la chose qui est souvent de loin la plus renforçante pour un être doué de raison: de réussir une organisation satisfaisante, une façon viable de traiter un certain secteur de l'expérience. Ce fait ajoute une dimension différente au concept de renforcement car quel que soit l'élément qui constitue dans ces cas un résultat plaisant, il est complètement engendré à l'intérieur même du système du sujet.

Le renforcement auto-engendré possède un potentiel énorme chez les

êtres cognitifs capables de réflexion. (Nous tous, j'en suis sûr, avons consacré du temps précieux et sué sur des casse-tête dont la solution n'apporta ni biscuits, ni argent, et une approbation sociale négligeable). Mais ce potentiel doit être développé et réalisé.

Quand des enfants commencent à jouer avec des blocs de bois, tôt ou tard ils les placent l'un par-dessus l'autre. Quelle que soit la satisfaction qu'ils éprouvent de la structure résultante, elle les stimule suffisamment pour les amener à répéter l'action et à l'améliorer. Ils peuvent, par exemple, se fixer implicitement ou explicitement comme objectif de construire une tour comprenant tous les blocs. S'ils réussissent, ils sont manifestement satisfaits, indépendamment des récompenses tangibles ou des commentaires d'adultes: la preuve en est qu'ils construisent aussi des tours en l'absence d'observateurs. La récompense provient de l'accomplissement, de l'imposition délibérée et réussie d'un ordre inhérent à leur propre façon d'organiser. Pour répéter l'exploit, la tour a besoin d'être démolie. Ceci s'avère aussi une source de satisfaction parce qu'encore une fois cela met en évidence le pouvoir de celui qui expérimente sur la structuration de son expérience.

Pour certains, ces observations peuvent paraître triviales. Pour moi, elles illustrent une caractéristique fondamentale du modèle de l'être doué de cognition, une caractéristique dont il faut tenir compte si l'on veut éduquer.

D'un point de vue constructiviste, il est insoutenable de supposer qu'une grande satisfaction sur le plan cognitif puisse surgir simplement parce que l'on nous a dit que nous avons bien fait quelque chose, du moment que l'exactitude est évaluée par quelqu'un d'autre. Pour devenir une source de réelle satisfaction, l'exactitude doit être vue comme l'ajustement à un ordre que nous avons établi nous-mêmes.

Les enseignants, aussi bien que les mathématiciens, sont portés à croire qu'objectivement dans tout cas particulier il existe un

problème, ainsi qu'une solution objectivement "juste". On s'attend donc à ce que les enfants et les étudiants de tous âges en viennent pour ainsi dire à "voir" le problème, sa solution, et les liens nécessaires unissant les deux. Mais la nécessité est d'ordre conceptuel et elle ne peut surgir de rien d'autre que de la prise de conscience des structures et opérations en jeu dans la conceptualisation, par le sujet, du problème et de sa solution. La nécessité logique ou mathématique ne réside pas dans quelque monde indépendant -- pour la voir et en retirer de la satisfaction, l'on doit réfléchir à ses propres constructions et à la façon dont on les a assemblées.

DERNIERES REMARQUES

Les enseignants ont un objectif commun, celui de générer la connaissance chez leurs élèves. Toutefois, dans la perspective épistémologique que j'ai esquissée, il semble que la connaissance ne soit pas une denrée transférable, ni la communication un véhicule.

Si, alors, nous en venons à voir la connaissance et la compétence comme des produits de l'organisation conceptuelle, par l'individu, de sa propre expérience, le rôle du maître ne consistera plus à répandre la "vérité" mais plutôt à aider et guider l'élève dans l'organisation conceptuelle de certains domaines d'expériences. Pour y parvenir, deux choses sont exigées du maître; d'une part, savoir passablement bien où est rendu l'étudiant, et d'autre part, connaître assez bien où il va. Aucune des deux n'est accessible par l'observation directe. Ce que dit en fait l'élève peut être interprété en termes d'un modèle hypothétique -- et c'est là en éducation, un domaine de recherche que, depuis Socrate, tout bon professeur a touché intuitivement. Aujourd'hui, nous sommes pas mal plus près du moment où nous pourrions fournir à l'enseignant un ensemble d'outils diagnostiques relativement fiables.

En ce qui a trait au rôle d'aide et de guide, les bons professeurs ont toujours trouvé des voies et des moyens de remplir ce rôle, car, consciemment ou inconsciemment, ils avaient réalisé que, bien qu'on puisse montrer le chemin à l'aide de mots et de symboles, il revient

à l'élève de conceptualiser et d'opérer.

Reste la destination, c'est-à-dire la façon d'opérer qui serait considérée "la bonne" du point de vue de l'expert. Comme je l'ai mentionné plus tôt, un modèle conceptuel de la formation des structures et des opérations constituant la compétence en mathématique est essentiel parce que, seul, un tel modèle pourrait indiquer dans quelle direction l'élève doit être guidé. La sorte d'analyse qui, toutefois, pourrait donner en détails la voie à suivre dans la construction de concepts mathématiques vient à peine d'être entamée. C'est dans ce domaine, qu'à mon point de vue, la recherche pourrait permettre de faire des progrès résultant en bénéfices immédiats pour la pratique de l'enseignement. Si en guidant l'élève l'objectif de l'enseignant est de faire naître la compréhension, plutôt que de l'entraîner à des habiletés spécifiques, sa tâche en sera certainement bien facilitée si cet objectif peut être représenté par un modèle explicite des concepts et des opérations que nous croyons être la source opératoire de la compétence mathématique. Encore plus important, si nous voulons que les élèves goûtent un peu de la satisfaction qu'éprouve le mathématicien à faire de la mathématique, on ne peut s'attendre à ce qu'ils la trouvent dans quelque récompense accordée pour leur performance, mais seulement dans la prise de conscience qu'ils ont bien ajusté à la réalité la construction de leurs propres schèmes conceptuels.

NOTES

- (1) Pour l'un d'eux, Parménide, le monde réel était un tout indivisible, statique; pour d'autres Leucippe et Démocrite, le monde réel était une masse d'atomes constamment en mouvement.
- (2) Un exposé excellent se trouve dans Popkin, R.H., The history of scepticism from Erasmus to Spinoza. Berkeley, Cal.: University of California Press, 1979

- (3) Berkeley a été interprété de cette façon par Popper (op. cit. chapitres 3 et 6). L'instrumentalisme de Vico est explicite dans son De antiquissima Italorum sapientia de 1710 (Naples: Stamperia de Classici, 1858). L'analyse de l'induction par Hume est instrumentaliste et le programme de Kant sur "l'enquête transcendantale dans le fonctionnement de la raison (Dans La Critique de la raison pure) fournit aux instrumentalistes modernes des munitions puissantes; cf. Mack, E., The guiding principles of my scientific theory of knowledge (1910), dans Toulmin, S., Physical reality (New York: Harper Torchbooks, 1970) et Simmel, G., Hauptprobleme der Philosophie (Berlin: de Gruyter, 7^{ème} édition, 1950).
- (4) Si cet isolement de la chose nommée ou "réfèrent" est une tâche exigeante avec des composés perceptuels relativement simples, tel que pomme, cela est évidemment beaucoup plus difficile lorsque le sens du mot est un concept qui exige une abstraction plus poussée à partir de l'expérience sensorielle ou des opérations mentales. Mais comme nous voulons continuer à soutenir que des mots tels que "tout" et "quelque", "le mien" et "le tien", "cause" et "effet", "espace" et "temps", et quantité d'autres ont une signification, nous devons supposer que ces significations, même si elles ne peuvent être directement perçues, sont quand même en quelque sorte isolées et rendues accessibles à toute personne apprenant la langue.
- (5) Pour des analyses conceptuelles récentes, voir Steffe, L.P., von Glasersfeld, E., Richards, J., & Cobb, P. Children's Counting types: Philosophy, theory and application. New York: Praeger, 1983

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B. REACTION PAPERS / REPLIQUES

EPISTEMOLOGY AND RESEARCH IN MATHEMATICS
EDUCATION: A REACTION

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Epistemological considerations never loomed large under the empiricist study of teaching and learning. In fact, the issue of knowledge per se never entered into the equation. What was important was observable behavior that could be measured and quantified. A major consequence of the constructivist revolution in psychology has been to put knowledge squarely into the picture in the study of learning.

LEARNING AS A CONSTRUCTIVE ACTIVITY

von Glasersfeld (1983) provides an historical/philosophical analysis of the foundations of a constructivist approach to the study of learning. He argues that the classical concept of knowledge or truth as a perfect match between an external reality and the knower's cognitive structures is fatally flawed. If experience is the only contact that the knower has with reality, there is no way to determine whether such a match exists.

von Glasersfeld argues that it is not necessary to assume the existence of an external reality. Reality is simply what we experience. We do not learn objective facts by discovering regularities in the world; we build cognitive structures based on invariants and consistencies in our experience. Facts are constructed by us, and they are viable as long as they are consistent with our experience and can be used to solve the problems we encounter.

von Glasersfeld also argues that words do not have an external content or meaning of their own. Each individual abstracts the meaning of words from his or her own experience. For communication, it is not necessary for individuals to share the same representations associated with a given word; it is sufficient that the speaker's representations are "compatible" with the listener's.

There is nothing particularly radical about von Glasersfeld's basic analysis of the nature of knowledge. It is generally consistent with prominent psycho-linguistic theories of communication, and constructivism has been widely embraced in psychology and mathematics education. I don't believe that the critical question at this time is whether knowledge is constructed. There seems to be fairly widespread agreement on that point, and the last remaining vestiges of behaviorism are not going to be swayed by metaphysical arguments.

The critical questions are what is the nature of the mathematical knowledge that is constructed and how is this knowledge constructed. Constructivism per se sets very few parameters on the kind of knowledge that is constructed. It is possible to accept the principle that knowledge is constructed and still regard mathematics as a collection of routine skills. Although many former behaviorists have been converted to cognitive scientists and tacetly accept the fundamental constructivist position, their basic conception of what represents mathematical knowledge has often not been significantly affected. Behavioral objectives have been replaced by cognitive objectives and empirical task analysis (Resnick & Ford, 1982). But the net effect has often been to reduce mathematics to a series of component skills that can be taught directly.

The nature of mathematical knowledge and how it is acquired cannot be derived directly from the principles of constructivism outlined by von Glasersfeld in the first part of his paper. For the most part, this basic theory provides relatively little explicit direction for the teaching of mathematics. Basic principles of constructivism certainly influence how we think and talk about teaching and learning. But an argument can be made that almost any form of instruction is at least not inconsistent with the idea that knowledge is constructed by the learner.

In the final section of his paper, von Glasersfeld deals with questions of the nature of knowledge and how it is acquired. For the most part, the discussion is quite general and would apply to

virtually any subject. But he does make several excellent points. One is a distinction he makes between formal mathematical definitions and mathematical concepts, a distinction that was generally overlooked in the curricular reforms of the 1960s. The basic principles of the teaching experiment are also sound: Have an adequate idea of where the student is and a "conceptual model of the structures and operations that constitute mathematical competence" (p.67). But very little direction is given as to how we "help and guide the student in the conceptual organization of certain areas of experience" (p.67).

I would like to close this discussion with one final comment that is not explicitly related to the paper or anything von Glasersfeld has said elsewhere. But it is a point that I think sometimes inhibits communication between researchers adopting constructivist orientations. Although we construct our own knowledge, we often act and speak as though our constructions represent an objective, shared reality. It is important to understand that an experimenter's "observations" are constructions as much as the conceptual structures constructed by the subject and to realize that problem situations do not have an inherent structure of their own; we impose structure on the problems based on our own mental constructs. But communication is not facilitated if we must always be as cautious with the use of language and as careful to acknowledge the constructivist nature of our own concepts as we are in posing metaphysical arguments.

EPISTEMOLOGY AND MATHEMATICAL KNOWLEDGE

Vergnaud (1983) focuses on a very different set of epistemological issues than von Glasersfeld (1983). He does not address the question of what is the general nature of knowledge. He is concerned with the structure of knowledge involved in the acquisition of specific "conceptual fields". Conceptual fields are bounded clusters of integrated concepts and situations. Vergnaud argues that children's learning of mathematics cannot be understood simply in terms of a psychological analysis of children's performance on a

set of mathematical tasks. Mathematical competence can only be understood in terms of the content to which it is applied. It is essential to analyze the mathematical nature of the conceptual field and attempt to understand children's developing concepts in light of this analysis.

Specifically, the analysis takes the following form:

To which problem or problems (within a conceptual field), does a new concept, a new property of a concept, a new procedure, or a new representation bring a solution or eventually a more powerful one? (Vergnaud, 1983, p.3)

Thus, the basic epistemological questions that Vergnaud is concerned with are what it means to know specific concepts within specific conceptual fields. Epistemological problems are basic complexities inherent in the mathematical structure of a conceptual field that make it difficult to acquire the concepts in the conceptual field.

I am in essential agreement with Vergnaud about the importance of analyzing the structure of conceptual fields. Although it does not encompass as broad a conceptual field, my own research on addition and subtraction is based on an analysis that parallels that proposed by Vergnaud (Carpenter & Moser, 1982). I would borrow a note from Von Glasersfeld to observe, however, that the structure of a conceptual field does not reside in the content itself. It is constructed by the researcher; it is not absolute, and alternative constructions are possible. The essential value of a particular construction must be gaged in terms of how well it accounts for children's performance. Thus, conceptual fields should not be constructed simply through a logical analysis of the mathematical structure of a problem domain. They should be constructed through a dynamic interplay between the researcher's analysis of his or her own conception of the mathematical structure of the conceptual field and children's concepts as represented by their solutions to problems in the conceptual field.

COMMON GROUND AND BASIC DIFFERENCES

Vergnaud (1983) proposes that: "Research in mathematics education studies the transmission [emphasis mine] and appropriation of mathematical knowledge" (p.2) and that "acquisition of a social knowledge like mathematics is not reducible to a process of spontaneous construction" (p. 2). At face value these quotes appear to be in conflict with von Glasersfeld's fundamental constructivist position. Vergnaud does not elaborate extensively on these themes so it is difficult to assess the extent of the disagreement. Von Glasersfeld would not use a term like "transmission of knowledge" but the difference may be more one of semantics than a basic philosophical schism.

The idea that mathematics cannot be reduced to "spontaneous construction" is not necessarily at odds with Von Glasersfeld's position. von Glasersfeld (1983) acknowledges that it is appropriate "to guide the child's conceptualizations toward adult competence" (p. 23), so the child's constructions are not entirely spontaneous.

The major theme of Vergnaud's paper deals with a problem that von Glasersfeld just touches on, the analysis of the structure of specific mathematical concepts. Von Glasersfeld deals with the problem in global terms. He acknowledges that the experimenter/teacher must have an "analytical model of the adult conceptualization towards which his guidance is to lead" (p.62). He goes on to say that "the structure of mathematical concepts is still largely obscure" (p.63). This is precisely the problem that Vergnaud addresses. In essence, what his paper does is to provide specific examples of the kind of analysis that von Glasersfeld proposes is essential.

At times in the past, individuals from von Glasersfeld's research team have questioned the appropriateness of the kind of analysis of mathematical structure proposed by Vergnaud. The essence of their arguments seemed to be based on the thesis that problems

themselves do not have an inherent mathematical structure. But as has been argued in this paper, the analysis of mathematical concepts does not necessarily assume that the structure is inherent in the concepts themselves. It can be regarded as the researcher's specification of his or her own constructs that are providing a basis for the researcher's construction of a model of the child's concepts. This analysis is entirely consistent with the basic philosophical position proposed by von Glasersfeld.

Thus, von Glasersfeld's basic constructivist position is sufficiently broad to encompass Vergnaud's analysis of conceptual fields. However, if one examines how they each operate in practice, there appear to be some fundamental differences between the conceptual analyses of mathematics structure that comes out of von Glasersfeld's research project (Steffe, von Glasersfeld, Richards, & Cobb, 1983) and those proposed by Vergnaud (1983). The analysis in von Glasersfeld's project is based primarily on children's performance with relatively little attention given to the problem space upon which the analysis is founded. For Vergnaud, the emphasis is on the analysis of the conceptual field, and children's performance is characterized on the basis of this analysis. Thus, in practice, Vergnaud and von Glasersfeld have begun from very different starting places. von Glasersfeld has initiated his analysis with the child's concepts; Vergnaud, with the conceptual field.

EPISTEMOLOGY AND INSTRUCTION IN MATHEMATICS

The ultimate question is what does all this mean for instruction in mathematics? On this question, the papers provide the least guidance. There seems to be a general consensus that we need to have a clear idea about what students know and a clear picture of the knowledge we want them to acquire, but very little direction is provided regarding how to guide students to attain that knowledge. In part, this seems to reflect the basic epistemological orientations of the authors. Mathematical knowledge cannot be reduced to a collection of isolated skills, and formulas for

instruction cannot be specified.

However, if we are unable or unwilling to provide more direction for instruction, we are in danger of conceding the curriculum to those whose basic epistemology allows them to be more directive. If we cannot provide clearer direction for instruction, instructional decisions are going to be made on practical grounds, and instructional practice is going to determine the epistemology that is reflected in the mathematics curriculum rather than the other way around.

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L'EPISTEMOLOGIE ET LA RECHERCHE EN DIDACTIQUE
DE LA MATHÉMATIQUE : UNE REPLIQUE
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Les considérations épistémologiques n'ont jamais occupé le premier plan dans l'étude empirique de l'enseignement et de l'apprentissage. De fait, la question de la connaissance en soi n'est jamais entrée en ligne de compte. La chose importante était le comportement observable pouvant être mesuré et quantifié. Une conséquence majeure de la révolution constructiviste en psychologie fut de placer carrément la connaissance au centre de nos préoccupations dans l'étude de l'apprentissage.

L'APPRENTISSAGE COMME ACTIVITE CONSTRUCTIVE

Von Glasersfeld (1983) fait une analyse historico-philosophique des fondements d'une approche constructiviste de l'étude de l'apprentissage. Il soutient que le concept classique de la connaissance ou de la vérité, en tant que correspondance parfaite entre la réalité extérieure et les structures cognitives de celui qui connaît, est fatalement vicié. Si l'expérience est le seul contact avec la réalité que possède celui qui connaît, il n'y a pas moyen de déterminer si une telle correspondance existe.

Von Glasersfeld soutient qu'il n'est pas nécessaire de présumer l'existence d'une réalité extérieure. La réalité est simplement ce que nous éprouvons. Nous n'apprenons pas de faits objectifs par la découverte de régularités dans le monde; nous construisons des structures cognitives à partir d'invariants et d'uniformités dans notre expérience. Les faits sont construits par nous, et ils sont viables en autant qu'ils sont compatibles avec notre expérience et qu'ils peuvent être utilisés pour résoudre les problèmes que nous rencontrons.

Von Glasersfeld prétend aussi que les mots ne possèdent pas de contenu externe ou de signification en soi. Chaque individu abstrait le

sens des mots à partir de sa propre expérience. Pour communiquer, il n'est pas nécessaire que les individus partagent les mêmes représentations associées à un mot donné; il suffit que les représentations du locuteur soient "compatibles" avec celles de l'auditeur.

Il n'y a rien de particulièrement révolutionnaire dans l'analyse de base que fait von Glasersfeld de la nature de la connaissance. Cette analyse est généralement en accord avec les théories psycho-linguistiques éminentes de la communication, et le constructivisme a été largement adapté en psychologie et en didactique de la mathématique. Je ne crois pas que, présentement, la question principale soit de savoir si la connaissance est construite. Il semble y avoir un large consensus sur ce point, et les derniers vestiges du behaviorisme ne seront pas éliminés par des arguments métaphysiques.

Les questions cruciales sont "quelle est la nature de la connaissance mathématique qui est construite?" et "comment cette connaissance est-elle construite?". Le constructivisme en soi donne très peu d'indications quant au type de connaissances qui sont construites. Il est possible d'accepter le principe que la connaissance se construit, tout en considérant la mathématique comme une collection d'habiletés routinières. Même si plusieurs personnes, autrefois behavioristes, se sont convertis pour devenir des scientifiques cognitifs et acceptent implicitement la position constructiviste fondamentale, souvent leur conception de base de ce que représente la connaissance mathématique n'en a pas été significativement affectée. Les objectifs behavioraux ont été remplacés par des objectifs cognitifs et l'analyse empirique de tâches (Resnick et Ford, 1982). Mais le résultat net a souvent été de réduire la mathématique à un ensemble de composantes, chacune étant considérée comme une habileté pouvant s'enseigner directement.

La nature de la connaissance mathématique et la façon dont elle est acquise ne peuvent être déduites directement des principes du constructivisme tels qu'ébauchés par von Glasersfeld dans la première partie de son exposé. Dans l'ensemble, cette théorie de base fournit

relativement peu de directives explicites pour l'enseignement de la mathématique. Les principes fondamentaux du constructivisme influencent certainement notre façon de penser et de parler de l'enseignement et de l'apprentissage. Mais l'on pourrait soutenir qu'à peu près toute forme d'enseignement n'est pas incompatible avec l'idée que la connaissance est construite par celui qui apprend.

Dans la dernière section de son texte, von Glasersfeld traite de questions touchant la nature de la connaissance et la façon dont elle est acquise. Pour une large part, la discussion est passablement générale et elle pourrait s'appliquer à n'importe quel sujet. Cependant, plusieurs de ses remarques sont excellentes. L'une d'elles est la distinction qu'il fait entre définition mathématique formelle et concept mathématique, distinction qui fut généralement oubliée dans les réformes de programmes des années soixante. Les principes de base de l'expérimentation didactique sont aussi sensés: "savoir de façon adéquate où se trouve l'étudiant" et "avoir un modèle conceptuel des structures et des opérations qui constituent la compétence mathématique" (p.98). Mais très peu de directives sont données sur la façon "d'aider et de guider l'étudiant dans l'organisation conceptuelle de certains domaines de l'expérience" (p.98).

J'aimerais clore cette discussion par un dernier commentaire qui n'est pas explicitement relié au texte de von Glasersfeld ou à ce qu'il a pu dire ailleurs. Mais c'est un point qui, je crois, empêche souvent la communication entre les chercheurs adoptant des orientations constructivistes. Même si nous construisons notre propre connaissance, souvent nous agissons et parlons comme si nos constructions représentaient une réalité objective et partagée. Il est important de comprendre que les "observations" d'un expérimentateur sont tout autant des constructions que les structures conceptuelles construites par le sujet, et de réaliser que des situations de problèmes ne possèdent pas une structure inhérente; nous imposons aux problèmes une structure basée sur nos propres construits mentaux. Cependant, la communication n'est pas facilitée si nous devons toujours prendre autant de précautions dans l'emploi du langage et être aussi prudent

à reconnaître la nature constructiviste de nos propres concepts que nous le sommes en avançant des arguments métaphysiques.

L'EPISTEMOLOGIE ET LA CONNAISSANCE MATHÉMATIQUE

Vergnaud (1983) se centre sur un ensemble de questions épistémologiques bien différentes de celles de von Glasersfeld (1983). Il ne s'attache pas à la question de la nature générale de la connaissance. Il se préoccupe de la structure de la connaissance impliquée dans l'acquisition de "champs conceptuels" spécifiques. Les champs conceptuels sont des amas de concepts intégrés et de situations. Vergnaud soutient que l'apprentissage de la mathématique par les enfants ne peut être compris simplement en termes d'une analyse psychologique de la performance des enfants dans un ensemble de tâches mathématiques. La compétence mathématique ne peut être comprise qu'en termes du contenu auquel elle est appliquée. Il est essentiel d'analyser la nature mathématique du champ conceptuel et de chercher à comprendre le développement des concepts chez les enfants à la lumière de cette analyse.

Plus spécifiquement, l'analyse prend la forme suivante:

Pour quel problème ou problèmes (à l'intérieur d'un champ conceptuel), un nouveau concept, une nouvelle propriété d'un concept, une nouvelle procédure, ou une nouvelle représentation apportent-ils une solution ou éventuellement une solution plus puissante? (Vergnaud, 1983, p.22)

Ainsi, les questions épistémologiques fondamentales qui préoccupent Vergnaud concernent le sens à donner à "connaître des concepts spécifiques dans le cadre de champs conceptuels spécifiques". Des problèmes épistémologiques sont des éléments fondamentaux de complexité inhérents à la structure mathématique d'un champ conceptuel qui rendent difficile l'acquisition de concepts dans ce champ conceptuel.

Je suis essentiellement d'accord avec Vergnaud en ce qui touche l'importance d'analyser la structure de champs conceptuels. Même si elle n'embrasse pas un champ conceptuel aussi vaste, ma propre recherche sur l'addition et la soustraction est basée sur une analyse qui suit

de près celle proposée par Vergnaud (Carpenter et Moser, 1982). Reprenant une réflexion de von Glasersfeld je ferais toutefois observer que la structure d'un champ conceptuel ne réside pas dans le contenu lui-même. Elle est construite par le chercheur; elle n'est pas absolue, et des constructions alternatives sont possibles. La valeur essentielle d'une construction particulière doit être jugée d'après sa capacité à rendre compte de la performance des enfants. Aussi, des champs conceptuels ne devraient pas être construits simplement à partir d'une analyse logique de la structure mathématique d'un domaine de problème. Ils devraient l'être par un échange dynamique entre l'analyse, par le chercheur, de sa propre conception de la structure mathématique du champ conceptuel et les concepts des enfants tels que représentés par leurs solutions à des problèmes dans le champ conceptuel.

TERRAIN D'ENTENTE ET DIFFERENCES FONDAMENTALES

Vergnaud (1983) propose que: "La recherche en didactique de la mathématique étudie la transmission [souligné par moi] et l'appropriation de la connaissance mathématique" (p.21) et soutient que "l'acquisition d'une connaissance sociale comme la mathématique ne peut se réduire à un processus de construction spontanée" (p.21). Prises à la lettre ces citations semblent entrer en conflit avec la position constructiviste fondamentale de von Glasersfeld. Vergnaud ne développe pas beaucoup ces thèmes, aussi est-il difficile d'évaluer l'importance du désaccord. Von Glasersfeld n'utiliserait pas une expression telle que "la transmission de connaissances" mais la différence peut relever plus de la sémantique que d'un schisme philosophique fondamental.

L'idée que la mathématique ne peut être réduite à une "construction spontanée" n'est pas nécessairement en contradiction avec la position de von Glasersfeld. Von Glasersfeld (1983) reconnaît qu'il est approprié "de guider l'enfant vers les conceptualisations de l'adulte" (p. 94), de sorte que les constructions de l'enfant ne sont pas entièrement spontanées.

Le thème majeur du texte de Vergnaud traite d'un problème que von Glasersfeld ne fait qu'effleurer, à savoir l'analyse de la structure de concepts mathématiques spécifiques. Von Glasersfeld traite du problème de façon globale. Il admet que l'expérimentateur/enseignant doit avoir "un modèle analytique de la conceptualisation adulte vers laquelle il doit guider l'étudiant" (p. 94). Il ajoute que "la structure des concepts mathématiques est encore en grande partie obscure" (p. 23). C'est précisément le problème auquel s'attaque Vergnaud. En somme, ce qu'il fait dans son texte, c'est de fournir des exemples spécifiques du type d'analyse qui, d'après von Glasersfeld, est essentiel.

A certains moments dans le passé, des membres du groupe de recherche de von Glasersfeld se sont questionnés sur la pertinence de la sorte d'analyse de structure mathématique proposée par Vergnaud. L'essentiel de leurs arguments semblait reposer sur la thèse voulant que des problèmes ne comportent pas de structure mathématique inhérente. Mais, tel qu'avancé dans ce texte, l'analyse de concepts mathématiques ne présume pas nécessairement que la structure est inhérente aux concepts eux-mêmes. Elle peut être entrevue comme la spécification, par le chercheur, de ses propres construits lesquels lui fournissent une base pour la construction d'un modèle des concepts de l'enfant. Cette analyse est entièrement compatible avec la position philosophique de base proposée par von Glasersfeld.

Donc, la position constructiviste de base de von Glasersfeld est suffisamment large pour englober l'analyse de champs conceptuels de Vergnaud. Cependant, si l'on examine comment chacun d'eux se comporte dans la pratique, il semble exister une différence fondamentale entre les analyses conceptuelles de la structure des mathématiques découlant du projet de recherche de von Glasersfeld (Steffe, von Glasersfeld, Richard et Cobb, 1983), et celles proposées par Vergnaud (1983). L'analyse présentée dans le projet de von Glasersfeld est principalement basée sur la performance des enfants, alors que relativement peu d'attention est accordée à l'espace du problème sur lequel l'analyse est fondée. Pour Vergnaud, l'accent est mis sur l'analyse du champ

conceptuel, et la performance des enfants est caractérisée à partir de cette analyse. Donc, en pratique, Vergnaud et von Glasersfeld ont adopté des points de départ bien différents. Von Glasersfeld a amorcé son analyse avec les concepts de l'enfant et Vergnaud, avec la notion de champ conceptuel.

L'EPISTEMOLOGIE ET L'ENSEIGNEMENT DE LA MATHÉMATIQUE

La question ultime c'est de savoir ce que tout cela signifie pour l'enseignement de la mathématique. Sur cette question, les textes ne fournissent guère de renseignements. Il semble y avoir accord général sur le fait que nous avons besoin de savoir clairement ce que les étudiants connaissent et quelles sont les connaissances que nous voulons leur voir acquérir, mais très peu de directives sont fournies sur la façon de les guider pour qu'ils parviennent à cette connaissance. Ceci semble refléter en partie les orientations épistémologiques de base des auteurs. La connaissance mathématique ne peut être réduite à une collection d'habiletés isolées, et des formules d'enseignement ne peuvent être précisées.

Toutefois, si nous sommes incapables ou si nous ne voulons pas fournir plus de directives pour l'enseignement, nous courons le risque de voir le curriculum confié aux gens défendant une épistémologie de base leur permettant d'être plus directifs. Si nous ne pouvons fournir de directives plus claires pour l'enseignement, les décisions touchant celui-ci seront prises à partir de raisons pratiques, et c'est la pratique de l'enseignement qui déterminera l'épistémologie reflétée dans le curriculum mathématique plutôt que l'inverse.

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KNOWING MATHEMATICS AND KNOWING RESEARCH

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No one ever derived a single item of new knowledge about learning from either epistemology or epistemology of education. What we know about reliable procedures by which learning may be furthered, we know through scientific or empirical psychology without benefit of epistemology.

--Sidney Hook, quoted by McClellan (1981, pp. 263-264)

L'observation scientifique est toujours une observation polémique.

--Gaston Bachelard (1934, p. 16)

Two questions arise when one adopts an epistemological perspective on research in mathematics education: (a) what is mathematical knowledge? and (b) what is research knowledge? The papers at hand (Vergnaud, 1983; von Glasersfeld, 1983) address the first question more explicitly than the second but raise issues connected with both.

The papers approach the question of mathematical knowledge from different angles, with von Glasersfeld attending primarily to the knowing person and Vergnaud to the matter known. One need not share von Glasersfeld's radical constructivism to appreciate his desire to understand how pupils interpret the mathematical tasks they are given in school. His analysis of the intrinsic motivation to be found in

fitting one's mathematical concepts together to achieve one's goals highlights a significant omission in the claim of the "new math" reformers that pupils would welcome the intellectual challenge of mathematical ideas: neither the ideas nor the problem of putting them together can be given to the pupil; they must be the pupil's own construction. Pupils do not so much learn mathematical structure as build mathematical structures. Vergnaud stresses the important point that since neither mathematical situations nor mathematical concepts can be adequately analysed in isolation, they must be studied as conceptual fields—a point that has implications both for teaching and for research. In his view, the main argument for an epistemological perspective is the need to describe mathematical competence in terms of its specific content. Whether or not one agrees with that view, one ought to see the value of explicating the "theorems in action" that pupils discover or invent.

The American comedian Johnny Carson is fond of getting laughs by giving an answer, asking the audience what the question is, and then supplying a question that is unexpected but somehow appropriate. Much of the humor depends on wildly diverse questions having the same answer. Von Glasersfeld and Vergnaud each urge researchers in mathematics education to follow Carson by examining the problem that a pupil's new knowledge solves. Here is what the pupil knows; to what problem was it a solution? The proposal is good advice. It is reminiscent of John Dewey's and George Kelly's view of the learner as hypothesis tester—to understand a learner's behavior try to understand what hypothesis he or she was testing. Recent work in

cognitive psychology suggests that this view can be quite fruitful, but--with Johnny Carson in mind--perhaps researchers ought to exercise some caution in working backwards from answer to question. In particular, any "model" a researcher constructs of a pupil's cognitions ought to be tested continually to see not only if it predicts the pupil's responses in a new situation but also if it fits other readings of the situations on which it was based.

What one considers mathematical knowledge is dependent in part on what one considers mathematics to be. Both authors are concerned with explicating the structure of mathematical concepts, but they offer somewhat different conceptions of mathematics itself. Von Glasersfeld emphasizes the numerical. He refers to "the art, the mystery, and the marvelous satisfaction of numerical operations" (p.44) and to "the discipline that is concerned with numbers, with arithmetic, and ultimately with mathematics" (p.41). His interest is clearly in the numerical side of elementary school mathematics. When he notes that units and sets can be created anywhere at any time and do not have an independent existence in an objective reality (p.62), one sees that his anti-Platonism extends to the foundations of mathematics. Vergnaud, on the other hand, takes a somewhat broader view of the discipline even while restricting his attention to school mathematics. He uses the nice figure of the building (p.3) to argue that epistemology of mathematics should not be limited to the foundations alone. If, as Davis and Hersh (1981) contend, "the typical working mathematician is a Platonist on weekdays and a formalist on Sundays" (p. 321), Vergnaud's characterization of

mathematics as "a functional answer to real problems" (p. 2) seems to make his view more typical than that of von Glasersfeld. Davis and Herah (pp. 400-405) offer the powerful metaphor of the building up of one's intuition about the four-dimensional hypercube through the manipulation of three-dimensional projections displayed on the screen of an interactive graphic system. They contend that something similar needs to happen in our view of mathematics. We must not "truncate mathematics to fit into a philosophy too small to accommodate it [but] rather . . . demand that the philosophical categories be enlarged to accept the reality of our mathematical experience" (p. 409). We can use various epistemological views like images on a computer screen to build up our understanding of what mathematics is, and it seems reasonable to extend the metaphor to our understanding of what mathematical knowledge is. On the other hand, one should not fall into an eclecticism that accepts all views uncritically. "Borrowing from many theories without carefully evaluating and incorporating the theory which renders intelligible that which is borrowed can only distort and falsify" (Petrie, 1972, p. 73).

Von Glasersfeld argues that meaning is not in the words a speaker or writer uses, it is in the listener's or reader's head. Perhaps meaning could be more usefully characterized as residing, from one point of view, in the words; from another point of view, in the heads; and from yet another point of view, in communication situations. The last point of view would imply, for instance, that the meaning of a statement might be affected by a speaker's intonation or a reader's presence in a group. Since so much of the communication that occurs

in school is in a setting involving many learners, it may be helpful to consider mathematical meaning at least some of the time as a social construction. The claim that understanding a communication is a matter of context forcing the modification of one's concepts, while important in reminding us of how we each construct our own interpretation of a communication, seems insufficient to account for the process of developing shared meaning. How are compatible representations achieved through such a process? Furthermore, one can certainly agree that educators have not given enough thought to linguistic communication in the teaching of mathematics without, however, agreeing that it has been taken for granted (von Glasersfeld, p. 43).

Von Glasersfeld's program of modeling each child's cognitive structure, modeling the adult's cognitive structure, and then guiding the child to transform the first into the second is likely to encounter practical problems of implementation all along the way. From the teacher's point of view, there is not simply the impracticality of finding time to construct a model for each of a class of children for each of a set of mathematical concepts--assuming the teacher knew how to do it--but there is also the question of how many different models a teacher can reasonably keep in mind as instruction is planned and pursued, given that tutorials are not likely to be feasible much of the time. The assumption that there is one adult cognitive structure--"the adult's way of adding and subtracting," say--not only seems something of an anomaly in a radical constructivist position but also raises questions as to how the

canonical adult structure is to be constructed. Beyond simply knowing this structure, the teacher needs to know how to portray it in some form to pupils so that they can make progress toward it. Guidance that may be difficult for a researcher to provide in a teaching interview is likely to be all but impossible for a teacher to provide in a classroom setting. Finally, there is the continuing problem that competing analyses of the concepts to be learned may be differentially effective in promoting the learning of mathematics. Easley (1983), for example, has suggested that the counting basis of the present curriculum--which Steffe, von Glasersfeld, and others have studied so extensively--may create difficulties for children that are avoided by the Japanese, who emphasize patterns and partitions in arithmetic.

As von Glasersfeld notes, the teaching of athletic skills illustrates the value of letting pupils see what they are doing at various stages of learning. As useful as this feedback is, recent analyses of skills such as juggling (Papert, 1980) and skiing (Fischer, Burton, & Brown, 1978) suggest that analyses of the teaching of these skills in terms of what has been called "increasingly complex microworlds" allows teacher and pupil to go beyond simple feedback to strategies of simplification and debugging. Although this approach has come from work in artificial intelligence, some debt is surely owed to behaviorism's insistence on the decomposition of performance into components.

Vergnaud's emphasis on the cognitive analysis of situations may be useful in attempting to understand better what adults and children do when they do mathematics. His view (p.19) of a concept as a

triple of situations (referents), invariants (signifieds), and symbolic representations (signifiers) provides a way of looking at the key problems of meaning and complexity he identified earlier in the paper (p. 2). Although he underplays the social dimension of meaning, he quite properly observes that meaning is conveyed by problematic situations that pupils perceive as real for them. His view of mathematical competence as best described by a partial order should be stressed in any consideration of how mathematical knowledge gets constructed and organized. There is, however, a potential "chicken-and-egg" problem connected with the cognitive analysis of situations. Vergnaud presents in his paper some pretty examples of such analyses. Where do the situations that he and his colleagues have examined come from? They appear to come primarily from school instruction, where they have been introduced precisely because someone once thought they had relevance for the teaching of some concept. But how can one know whether the essential invariants for a concept have been identified if one begins with school tasks? Where is it written that school tasks span the set of concept invariants? So does one begin with the concept or with the situation?

It appears that a cognitive analysis of situations must also be a cognitive analysis of concepts and of their symbolic representations. Vergnaud's notion of conceptual fields also suggests that such analyses must be conducted for sets of concepts and their associated situations and representations rather than being done one at a time. If one examines the burgeoning literature on young children's addition and subtraction, one can see there that research has helped to

identify situations that might not have appeared "naturally" in school instruction. This is one contribution that empirical research can make to an epistemological analysis, and it leads to the question of what contribution an epistemological perspective can make to research.

Von Glasersfeld is certainly correct in his observation that there is widespread disillusionment today concerning many aspects of education. The public's disillusionment with the perceived quality of the school graduates' knowledge, the teachers' disillusionment with various educational reforms promoted as panaceas, and the researchers' disillusionment with traditional paradigms for conducting and reporting research probably share some of the same causes, but they do not all arise from the same source. Moreover, it is unlikely that a re-examination of what it is "to know"--welcome though that might be--will be equally beneficial to all. How can it benefit research in mathematics education?

Let us examine first the research problems suggested by an epistemological perspective and then the research methods it might suggest.

The constructivist's contention that children construct their own knowledge structures casts the teacher in a new role that has yet to be examined critically in the research literature. If, indeed, the teacher follows what Polya (1981) terms "a classical Socratic precept," namely that "the ideas should be born in the students' mind and the teacher should act only as midwife" (vol. 2, p. 104), then

the skills of midwifery deserve careful study.

As has already been noted, the cognitive analysis of situations demands much more reliable knowledge than we now possess about the interplay between situations and concepts. Which situations give rise to which conceptions or misconceptions? Which situations, although not now present in the curriculum, illuminate the invariants of a concept? How does a given concept take on different meanings when embedded in different conceptual fields? How do different symbolic representations illuminate the invariants of a concept, and with which situations are they most productively associated? Facets of these questions are being investigated, but they take on a different appearance when seen through the lens provided by Vergnaud.

Empirical research can help in the process of explicating the structure of mathematical concepts. Mathematical knowledge concerns more than simply the ability to find new solutions to problems--although that is important. Recent studies of expert problem solvers and computer simulations of problem-solving processes have shown how the solution of a complex problem depends not only on procedures for representing and transforming the problem and a control system to guide the use of these procedures but also on a rich store of organized knowledge in the content domain of the problem. If, as Vergnaud asserts, mathematical knowledge develops as an adaptive response to problematic situations, much research is needed into the organized knowledge--as well as the procedures--required when concepts are developed through the solution of problems.

Knorr (1981) has argued that a constructivist approach to knowledge requires sensitive rather than frigid research methodologies. Sensitive methodologies, she claims, are marked by intersubjectivity (through close observation at the site of production), a decentered constructivity (that allows the field of study to constrain the information construed), and interactionism (so that the focus is no more upon individuals than upon society). The teaching interview (termed "teaching experiment") of Steffe and his colleagues has replaced the presumed neutrality of traditional research by an appropriate intersubjectivity--although one always needs to ask whether other sites might profitably be explored as well. Such interviews have not, however, entirely solved the problem of decentering the construction of data. As Knorr puts it, the problem "is not just to understand, but to let speak." (p. 18). Researchers need to translate as many as possible of their categories for constructing behavior into the categories that pupils and teachers employ. Further, the interactionism feature of a sensitive methodology holds that the explication of human action cannot simply be built up from a knowledge of individual actor's cognitions and that some attention must be given to the social context of action. A sensitive methodology would look at teaching and learning as they occur in the habitat of the classroom.

An epistemological perspective on research in mathematics education reinforces the value of an interdisciplinary approach to such research. Needed are mathematicians who understand the epistemology of mathematics, philosophers who understand the

epistemology of education, and research methodologists who understand the epistemology of research. Needed perhaps even more are mathematicians, psychologists, sociologists, anthropologists, and researchers in mathematics education who understand something of the learning and teaching of mathematics, and most of all, sensitive, gifted teachers of mathematics.

The question "what is research knowledge?" is most meaningful when one asks it with respect to the knowledge of research possessed by practitioners. What is their knowledge of research, and how can it be made more extensive and effective? What does it mean to know what research says about the teaching and learning of mathematics? How does that knowledge develop, and what is "adult" knowledge in this domain? The epistemology of research knowledge in mathematics education is an unstudied problem.

"Knowledge is not a transferable commodity," says von Glasersfeld (p.66). "Research in mathematics education studies . . . the transmission . . . of mathematical knowledge," says Vergnaud (p.2). So perhaps mathematical knowledge is not piped out like oil but rather broadcast like a television program; it is transmitted, not transferred. To oversimplify, an epistemological perspective can provide us with some metaphors with which to view our world. In doing such an analysis, we may see some things we did not see before--which is also supposed to be one of the benefits of research. Together perhaps epistemology and empirical research can contribute more to the improvement of mathematics education than either has done alone. The two papers at hand may have sparked a beginning.

Note

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CONNAITRE LA MATHÉMATIQUE ET CONNAITRE LA RECHERCHE

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Personne n'a jamais pu obtenir une seule parcelle de connaissance nouvelle à propos de l'apprentissage à partir, soit de l'épistémologie, ou soit de l'épistémologie de l'éducation. Ce que nous connaissons à propos de procédures fiables, par lesquelles l'apprentissage peut être poursuivi, nous le connaissons au moyen de la psychologie scientifique ou empirique, sans aucun apport de l'épistémologie.

- Sidney Hook, cité par McClellan (1981, pp. 263-264). (Traduction libre).

L'observation scientifique est toujours une observation polémique.

- Gaston Bachelard (1934, p. 16).

Lorsque l'on adopte une perspective épistémologique en recherche sur l'enseignement de la mathématique, deux questions se posent : (a) qu'est-ce que la connaissance mathématique? et (b) qu'est-ce que la connaissance de la recherche? Les textes présentés (Vergnaud, 1983; von Glasersfeld, 1983) traitent plus explicitement de la première question que de la seconde, mais ils soulèvent des points reliés aux deux.

Dans ces exposés la question de la connaissance mathématique est examinée sous différents angles : von Glasersfeld traite principalement de la personne qui connaît, et Vergnaud de la matière connue. Il n'est pas besoin d'endosser l'idée du constructivisme radical de von Glasersfeld pour pouvoir apprécier son désir de comprendre comment les élèves interprètent les tâches mathématiques qui leur sont présentées à l'école. Son analyse de la motivation intrinsèque, résultant de la mise en place par une personne de ses propres concepts mathématiques en vue de l'atteinte de ses objectifs, met en lumière un oubli significatif dans la prétention des réformateurs de la "nouvelle mathématique" que les élèves surveilleraient

avec joie le défi intellectuel posé par les idées mathématiques : ni les idées, ni le problème de les assembler ne peuvent être donnés aux élèves; ils doivent être construits par l'étudiant lui-même. Les élèves n'apprennent pas tant une structure mathématique qu'ils construisent des structures mathématiques. Vergnaud souligne le point important que, puisque ni les situations mathématiques, ni les concepts mathématiques ne peuvent être analysés adéquatement de façon isolée, ils doivent être étudiés en tant que champs conceptuels - un point qui a des implications à la fois pour l'enseignement et la recherche. Selon lui, l'argument principal en faveur d'une perspective épistémologique, c'est le besoin de décrire la compétence mathématique en termes de son contenu spécifique. Que l'on soit d'accord ou non avec ce point de vue, il faut reconnaître la valeur de pouvoir expliquer les "théorèmes-en-acte" que les élèves découvrent ou inventent.

Le comédien Johnny Carson prend plaisir à susciter le rire en donnant une réponse, puis en demandant à l'auditoire quelle est la question, et en fournissant à la fin une question inattendue mais en quelque sorte appropriée. Une grande partie de son humour dépend de questions les plus diverses possédant la même réponse. Von Glasersfeld et Vergnaud incitent tous deux les chercheurs en didactique de la mathématique, à imiter Carson en examinant le problème que permet de résoudre une nouvelle connaissance de l'élève. Voici ce que connaît l'élève; de quel problème était-ce une solution? Cette suggestion s'avère intéressante. Elle nous rappelle la façon qu'avaient John Dewey et George Kelly de voir celui qui apprend comme un vérificateur d'hypothèses - pour comprendre le comportement de celui qui apprend, tâche de comprendre quelle hypothèse il était en train de vérifier. Des recherches récentes en psychologie cognitive portent à croire que cette façon de voir peut être passablement bénéfique, mais - avec Johnny Carson en tête - peut-être les chercheurs devraient-ils faire montre de prudence lorsqu'ils travaillent de façon rétrograde, c'est-à-dire de la réponse à la question. En particulier, tout "modèle" construit par

un chercheur au sujet de la cognition d'un élève devrait continuellement être testé : pour voir, non seulement s'il prédit les réponses de l'élève dans une situation nouvelle, mais s'il intègre d'autres interprétations des situations sur lesquelles il était basé.

Ce que l'on considère être une connaissance mathématique, dépend en partie de notre vision de la nature même de la mathématique. Les deux auteurs sont avides d'expliquer la structure des concepts mathématiques, mais ils témoignent de conceptions quelque peu différentes de la nature de la mathématique elle-même. Von Glasersfeld met l'accent sur l'aspect numérique. Il parle de "l'art, du mystère, et de la satisfaction merveilleuse des opérations numériques" (p.73), et de "la discipline qui est concernée par les nombres, par l'arithmétique, et éventuellement par la mathématique" (p.70). Son intérêt penche clairement du côté numérique de la mathématique de l'école élémentaire. Lorsqu'il remarque que les unités et les ensembles peuvent être créés n'importe où, en tout temps, et qu'ils ne possèdent pas d'existence indépendante dans une réalité objective (p.93), l'on peut voir que son anti-platonisme rejoint les fondements de la mathématique. Vergnaud, par contre, adopte un point de vue passablement plus large de la discipline, même lorsqu'il se restreint à la mathématique de l'école. Il utilise la belle image de l'édifice (p.22) pour soutenir que l'épistémologie de la mathématique ne devrait pas se limiter aux fondements seuls. Si, comme Davis et Hersh (1981) le prétendent, "le mathématicien typique, dans son travail, est un platonicien sur semaine et un formaliste le dimanche" (p. 321), la caractérisation de la mathématique, par Vergnaud, comme étant "une réponse fonctionnelle à des problèmes réels" (p.23) semble rendre son point de vue plus conforme au travail réel du mathématicien que celui de von Glasersfeld. Davis et Hersh (pp. 400-405) proposent la métaphore du développement de notre intuition de l'hypercube dans la quatrième dimension, à l'aide de la manipulation de projections tridimensionnelles déployées sur l'écran d'un système graphique interactif. Ils prétendent que quelque chose de semblable doit se produire pour notre

vision de la mathématique. Nous ne devons pas "tronquer la mathématique pour qu'elle puisse prendre place dans une philosophie trop étroite pour l'accommoder (mais) plutôt ... exiger que les catégories philosophiques soient élargies pour accepter la réalité de notre expérience mathématique" (p. 409). Nous pouvons utiliser diverses approches épistémologiques, tel des images sur un écran d'ordinateur pour élaborer notre compréhension de ce qu'est la mathématique, et il semble raisonnable d'étendre la métaphore à notre compréhension de ce qu'est la connaissance mathématique. D'autre part, il ne faudrait pas tomber dans un éclectisme qui accepte tous les points de vue de façon non critique". Emprunter de plusieurs théories sans évaluer avec soin et sans incorporer la théorie qui rend intelligible ce qui est emprunté ne peut que produire de la distorsion et falsifier". (Petrie, 1972, p. 73).

Von Glasersfeld soutient que la signification n'est pas dans les mots utilisés par un locuteur ou un écrivain, mais plutôt dans la tête de l'auditeur ou du lecteur. Peut-être pourrait-on caractériser de façon plus utile la signification comme résidant, d'un certain point de vue, dans les mots; d'un autre point de vue, dans la tête; et encore d'un autre point de vue, dans les situations de communication. Le dernier point de vue impliquerait, par exemple, que la signification d'une proposition puisse être affectée par l'intonation du locuteur ou l'insertion du lecteur dans un groupe. Puisqu'une si grande part de la communication qui se produit à l'école a lieu dans une situation impliquant plusieurs apprenants, il peut être utile de considérer la signification mathématique, du moins en certaines occasions, comme une construction sociale. Bien qu'important pour nous de rappeler la façon dont chacun de nous construit sa propre interprétation d'une communication, la croyance que comprendre une communication est affaire de contexte, forçant ainsi l'individu à modifier ses propres concepts, semble insuffisante pour expliquer le processus de développement d'une signification qui soit partagée par plusieurs personnes. Comment des représentations compatibles sont-elles réalisées par un tel processus?

De plus on pourra certainement admettre que les enseignants n'ont pas accordé suffisamment d'attention à la communication linguistique dans l'enseignement de la mathématique sans aller jusqu'à croire que cela allait de soi. (Von Glasersfeld, p.73).

Le programme de von Glasersfeld concernant la modélisation de la structure cognitive de chaque enfant, la modélisation de la structure cognitive de l'adulte, ainsi que la façon de guider l'enfant pour transformer le premier au second, est susceptible de se buter continuellement à des problèmes pratiques d'implantation. Du point de vue de l'enseignant, non seulement y a-t-il la difficulté à construire un modèle pour chacun des élèves d'une classe et pour chacun d'un ensemble de concepts mathématiques -- en supposant que l'enseignant sache comment le faire -- mais il y a aussi la question du nombre de modèles différents qu'un enseignant peut raisonnablement garder en tête pendant la planification et le déroulement de l'enseignement, étant entendu que la formule du tutorat n'est praticable que pour une petite partie du temps. La supposition qu'il existe une structure cognitive adulte unique -- disons "la façon adulte d'additionner et de soustraire" -- non seulement semble en quelque sorte une anomalie dans une position constructiviste radicale, mais elle soulève aussi des interrogations sur la façon dont la structure canonique adulte doit être construite. Plus que de simplement connaître cette structure, l'enseignant doit savoir comment la présenter aux enfants sous une forme telle qu'ils puissent éventuellement l'atteindre. Comme l'aide à apporter durant une interview d'enseignement peut s'avérer difficile, même pour le chercheur, comment espérer qu'elle puisse être réalisée par l'enseignant dans les conditions régulières de la classe. Finalement, il y a toujours présent le problème posé par le fait que des analyses opposées des concepts à apprendre peuvent se révéler différemment efficaces pour la promotion de l'apprentissage de la mathématique. Easley (1983), par exemple, a émis l'opinion que la comptage, formant la base de notre présent curriculum, -- que Steffe, von Glasersfeld et d'autres ont étudié de façon si poussée -- peut

créer chez les enfants des difficultés que les Japonais semblent éviter du fait, qu'en arithmétique, ils mettent plutôt l'accent sur les patterns et les partitions.

Comme le fait remarquer von Glasersfeld, l'enseignement d'habiletés athlétiques illustre la valeur qu'il y a à permettre aux élèves de voir ce qu'ils font à différentes étapes de leur apprentissage. Quelqu'utile que cette rétroaction puisse être, des analyses récentes d'habiletés, tel que le jonglage (Papert, 1980) et le ski (Fisher, Burton et Brown, 1978) suggèrent que les analyses de l'enseignement de ces habiletés, en termes de ce qui a été appelé "des microcosmes de plus en plus complexes", permettent à l'enseignant et à l'élève de dépasser la simple rétroaction aux stratégies de simplification et de "debugging". Même si cette approche découle de travaux en intelligence artificielle, une certaine part du crédit doit certainement être rendue au behaviorisme, vu son insistance à toujours vouloir décomposer la performance en ses composantes.

L'importance qu'attache Vergnaud à l'analyse cognitive des situations peut être utile à toute tentative visant à mieux comprendre ce que font les adultes et les enfants lorsqu'ils font des mathématiques. Sa perception (p. 39) d'un concept, comme étant un triplet regroupant des situations (les référents), les invariants (les signifiés), et des représentations symboliques (les signifiants), fournit une façon d'examiner les problèmes fondamentaux de signification et de complexité qu'il a identifiés au préalable dans sa communication (p. 21). Bien qu'il n'insiste pas sur la dimension sociale de la signification, il observe très justement que le sens est transmis par des situations problématiques que les élèves perçoivent comme réelles. Sa perception de la compétence mathématique, comme étant au mieux décrite par un ordre partiel, devrait être soulignée dans toute considération de la façon dont la connaissance mathématique est construite et organisée. Il y a là cependant la possibilité de retomber dans le problème de "l'oeuf ou de la poule", relié à l'analyse cognitive des situations. Vergnaud présente dans son document quelques beaux exemples de telles

analyses. D'où viennent les situations que lui et ses collègues ont examinées? Elles paraissent provenir essentiellement de l'instruction scolaire dans laquelle elles ont été introduites, précisément parce que quelqu'un a cru qu'elles étaient pertinentes pour l'enseignement de certains concepts.

Mais comment peut-on savoir si les invariants essentiels d'un concept donné ont été identifiés si l'on commence par des tâches scolaires? Où donc est-il écrit que les tâches scolaires recouvrent l'ensemble des invariants d'un concept? Alors, faut-il commencer par le concept ou par la situation?

Il semble donc qu'une analyse cognitive des situations se doit aussi d'être une analyse cognitive de concepts et de leurs représentations symboliques. La notion de champs conceptuels de Vergnaud suggère aussi que de telles analyses doivent porter sur des ensembles de concepts, avec les situations et les représentations qui leur sont associées, plutôt que sur un seul concept à la fois. Si nous examinons le volume toujours croissant de la littérature sur l'addition et la soustraction chez les jeunes enfants, on s'aperçoit que la recherche a aidé à identifier des situations qui ne seraient peut être pas apparues "naturellement" dans l'enseignement scolaire. C'est là une contribution que la recherche empirique peut apporter à l'analyse épistémologique et cela nous amène à la question de savoir quelle contribution une perspective épistémologique peut apporter à la recherche.

Von Glasersfeld a certainement raison lorsqu'il observe qu'il existe aujourd'hui un grand désenchantement par rapport à plusieurs aspects de l'éducation. Le désenchantement du public quant à sa perception de la qualité des connaissances de nos diplômés, la perte des illusions des enseignants quant à certaines réformes en éducation, proposées comme des panacées, et le désillusionnement qu'éprouvent les chercheurs face aux paradigmes traditionnels se rapportant à la conduite et à la communication de la recherche, ont probablement des causes communes, mais ne remontent pas tous à la même source. De

plus, il est douteux qu'un réexamen de ce que c'est que "connaître" - aussi bienvenue que cela puisse être - sera également bénéfique à tout le monde. Comment cela peut-il être bénéfique à la recherche en didactique de la mathématique?

Examinons d'abord les problèmes de recherche suggérés par une perspective épistémologique et ensuite les méthodes de recherche qu'elle peut suggérer.

L'assertion des constructivistes à l'effet que les enfants construisent leurs propres structures de connaissances assigne à l'enseignant un rôle nouveau qu'il reste encore à examiner d'un oeil critique dans de nouvelles recherches. Si vraiment le maître croit à ce que Polya (1981) appelle "le précepte socratique classique" voulant que "les idées naissent dans l'esprit des élèves, le maître ne devant agir que comme sage-femme" (Vol. 2, p. 104), alors les habiletés de sages-femmes méritent une étude attentive.

Tel que mentionné auparavant, l'analyse cognitive des situations exige une connaissance beaucoup plus fiable que celle que nous possédons actuellement de l'interaction entre situations et concepts. Quelles situations suscitent quelles conceptions, justes ou fausses? Quelles situations, même encore absentes du curriculum, éclairent les invariants d'un concept? Comment un concept donné prend-il des sens différents lorsque enchâssé dans différents champs conceptuels? Comment différentes représentations symboliques éclairent-elles les invariants d'un concept et avec quelles situations est-il plus productif de les associer? Différents aspects de ces questions sont présentement étudiés, mais ils apparaissent de façon bien différente lorsque vus à travers la lentille que nous donne Vergnaud.

La recherche empirique peut contribuer à l'explication de la structure de concepts mathématiques. La connaissance mathématique implique plus que la simple habileté à trouver de nouvelles solutions aux problèmes - bien que cela soit important. Des recherches récentes sur des experts en résolution de problèmes et sur la simulation par

ordinateur des processus de résolution de problèmes ont montré comment la solution d'un problème complexe dépend, non seulement des procédures pour représenter et transformer le problème et du système de contrôle guidant l'utilisation de ces procédures, mais aussi d'une riche provision de connaissances organisées touchant le contenu du problème. Si, comme l'affirme Vergnaud, la connaissance mathématique se développe en tant que réponse adaptative à des situations problématiques, beaucoup de recherches sont nécessaires sur l'organisation de la connaissance et des procédures requises lorsque les concepts sont développés à travers la résolution de problèmes.

D'après Knorr (1981) l'approche constructiviste dans l'étude de la connaissance exige des méthodologies de recherche plutôt sensibles que frigides. Elle prétend que des méthodologies sensibles sont caractérisées par l'intersubjectivité (en observant de près au lieu même de la production), une constructivité décentrée (qui permet au champ d'étude de circonscrire l'information analysée), et l'interactionnisme (de façon à ce que le centre d'attention ne soit pas plus les individus que la société). L'entrevue didactique (appelée "expérimentation didactique") de Steffe et de ses collègues a remplacé la présumée neutralité traditionnelle de la recherche par une intersubjectivité appropriée -- bien qu'il faut toujours se demander si d'autres avenues ne pourraient pas tout aussi bien être explorées avec profit. De telles entrevues n'ont cependant pas résolu complètement le problème de décentration dans la construction de données. Tel que l'indique Knorr, le problème "n'est pas simplement de comprendre, mais de laisser parler". (p.). Les chercheurs doivent traduire autant de leurs catégories que possible, pour la construction du comportement, en des catégories que les élèves et les enseignants utilisent. De plus, le caractère interactif d'une méthodologie sensible implique que l'explication de l'action humaine ne peut pas être simplement construite à partir d'une connaissance de la cognition d'acteurs individuels et qu'une certaine attention doit être accordée au contexte social de l'action. Une méthodologie

sensible considérerait l'enseignement et l'apprentissage pendant qu'ils ont lieu dans le cadre de la classe.

Une perspective épistémologique de la recherche en didactique de la mathématique donne plus de valeur à l'approche interdisciplinaire pour de telles recherches. Ce dont nous avons besoin, c'est de mathématiciens qui comprennent l'épistémologie des mathématiques, de philosophes qui comprennent l'épistémologie de l'éducation, et de spécialistes en méthodologie de la recherche qui comprennent l'épistémologie de la recherche. Et peut-être avons-nous encore un plus grand besoin de mathématiciens, de psychologues, de sociologues, d'anthropologues et de chercheurs en didactique des mathématiques qui comprennent quelque chose à l'apprentissage et à l'enseignement des mathématiques, et encore plus d'enseignants de mathématiques doués et sensibles.

La question "qu'est-ce que la connaissance de la recherche?" a le plus de sens lorsqu'elle est posée au sujet de la connaissance de la recherche que possèdent les praticiens. Quelle est leur connaissance de la recherche et comment peut-elle être élargie et rendue plus efficace? Que signifie "savoir ce que la recherche dit au sujet de l'enseignement et de l'apprentissage des mathématiques?" Comment cette connaissance se développe-t-elle et qu'est-ce que la connaissance "adulte" en ce domaine? L'épistémologie de la connaissance de la recherche en didactique des mathématiques est un problème qui n'est pas étudié.

"La connaissance n'est pas une denrée transférable" dit von Glasersfeld (p.98). "La recherche en didactique des mathématiques étudie ... la transmission ... de la connaissance mathématique" dit Vergnaud (p.). Alors peut être que la connaissance mathématique n'est pas siphonnée comme de l'huile mais plutôt diffusée comme un programme de télévision; elle est transmise, non transférée. Pour simplifier à outrance, une perspective épistémologique peut nous pourvoir de certaines métaphores nous permettant d'observer

le monde. En faisant une telle analyse, nous pouvons voir certaines choses que nous n'avions pas vues auparavant -- ce qui est sensé être un des avantages de la recherche. Peut-être qu'ensemble, l'épistémologie et la recherche empirique peuvent contribuer plus à l'amélioration de la didactique des mathématiques qu'aucune, seule, n'a pu le faire. Les deux documents présents ont peut-être provoqué l'étincelle de départ.

NOTE

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C. ALGEBRA / ALGEBRE

SIGNS OF PROGRESS SOLVING ELEMENTARY ALGEBRAIC EQUATIONS

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RESUME

Il s'agit ici de la suite de travaux de l'auteur et d'autres chercheurs sur le développement des capacités de résolution des équations. Les travaux reposent sur des notions générales en ce qui concerne la résolution de problèmes et la psychologie vue comme un traitement de l'information. Il y a deux divisions principales: la première adapte les idées de Polya (1957) émises dans "SIGNS OF PROGRESS" à la résolution d'équations et la seconde discute des fonctions d'évaluation pour les équations (Wickelgren, 1974). Les résultats de classement d'équations-échantillons effectués par des élèves sont également présentés et comparés aux résultats de fonctions d'évaluation choisies. Des questions sur les bases psychologiques de telles fonctions sont ensuite posées.

INTRODUCTION

Recent work by the author and others (Bernard, 1978; Carry, Lewis, and Bernard, 1979; Bernard and Bright, 1983) speak to certain problem-solving aspects in the development of equation solving skill for the type of elementary equations commonly found in secondary school algebra courses, basic and advanced. In these works, each equation is associated with a state-action search space (Newell and Simon, 1972; Wickelgren, 1974) which provides a backdrop for examining both correct and incorrect performance for a specified population of solvers. Bernard (1978) explains how this approach is consistent with the school mathematics characterization of equation solving (see also Bernard and Bright, 1983) and the interests and needs of the mathematics education community. As a matter of practicality, a first approximation to the state-action space for a given equation might be constructed empirically by fusing together the written work of many solvers and their different attempts. This, however, would need to be supplemented and enriched using certain constructs found in theories of cognitive psychology: certainly those dealing with perception (Bernard, 1983), transformational thoughts (Bernard, 1978), operations, processes, strategies, and levels of cognitive control (Bernard and Bright, 1983). Of particular interest at this time are those higher levels of control and awareness which guide a solver through the search space together with the visible cues which might be used to evaluate and measure progress toward the goal.

In setting the stage for the conference discussion, this paper is in two parts. First, Polya's ideas about SIGNS OF PROGRESS, although originally expressed in a more general context and at a level leaving much to be operationalized, are reviewed and adjusted to the context of equation solving. The appropriateness of these adjustments and/or interpretations, then, will be one point of discussion.

The latter portion of this paper presents Wickelgren's notions of evaluation function as they might apply to equation solving (see Wickelgren, 1974). The results of students ranking a set of equations in order of difficulty will also be presented. Comparisons are then made between student rankings and those from selected evaluation

functions, examples included. Possible psychological bases for certain formal evaluation functions should provide the second point for discussion.

POLYA'S VIEWS, WITH ADJUSTMENTS

Polya (1957) states, "There are certain mental operations typically useful in solving problems ... If such a typical operation succeeds ... its success is felt as a sign of progress." The preceding is expressed in terms of his LIST. Thus, one should interpret his comments in relation to UNDERSTANDING THE PROBLEM, DEVISING A PLAN, CARRYING OUT THE PLAN, and LOOKING BACK, and in the present context, make suitable adjustments to the task of solving elementary algebraic equations.

Understanding the Problem: An algebraic equation is itself a condition to be satisfied. Progress is attained as the solver identifies the unknown, the nature of the solution (e.g., " $x = \dots$ " to some constant or algebraic expression not containing x), or restrictions or impossibilities (e.g., $x \neq 0$ in $(3/x) = x + 2$). In this case, awareness and feelings of progress will vary over solvers and equations as such realizations occur more or less automatically, if at all, and in no particular order.

Devising a Plan might entail surveying the features of the equation (unknowns on both sides or not, parentheses, fractions, quadratic expressions,...) and singling out those characteristics which are to receive the initial transformational efforts. Deciding which method and strategies to use could be difficult depending upon previous experiences. If the current equation is a perfect analog of one previously solved, and this is recognized by the solver, the solution action sequence from the earlier might be transferred and applied directly to the current equation. On the other hand, the current equation could involve new features or a novel composition of familiar features, thus calling for more effort and originality on the part of the solver in constructing a solution action sequence.

In this category, Polya also hints at progress made through synthesis of elements in the problem and those associated ideas which might occur to the solver. He explicitly comments about seeing relationships and bridging the gap between the givens and the goal. Thus, such things as seeing " $7 = 3 + 4$ " in connection with the equation $7(4x - 1) = 3(4x - 1) + 4$, implying that $4x - 1 = 1$, and noting that "zero works", referring to $(40 - x)$ in the equation $9(40-x)=5(40-x)$, illustrate his comments in the context of equation solving. It should be pointed out, however, that seeking and exploiting such relationships is never included as a topic in textbooks. Then too, the Carry, Lewis, and Bernard (1979) study found such exploitations to be rather rare, but existent. Polya may therefore be indicating an aspect of equation solving which needs further investigation.

On the other hand, there is a certain aspect of equation solving that might suggest a need for extending Polya's general theories. A large portion of equation solving is characterized in terms of escape from antigoals. Most of the symbols and structures in a given equation are considered unwanted complications, things to "GET RID OF". There is a sense of relief, hence progress, in moving from a more to less complicated state (Skemp, 1979). This view of movement away from the unwanted is as prevalent in equation solving as that of movement toward the goal and, because of the psychological distinctions between the two, ought to be the object of formal analyses in both equation solving and more general problem solving.

Carrying Out the Plan: One should be cautious not to assume that the two phases, Devising a Plan and Carrying Out the Plan, happen sequentially. Solvers observed in the act, usually explore various possibilities. Often, part of what might be considered a plan is carried out, the results evaluated, new directions considered, and adjustments sometimes made (Newell and Simon, 1972; Carry, Lewis, and Bernard, 1979; Bernard and Bright, 1983). In some cases, therefore, it is only after the final result has been attained that a solver might be able to indicate a complete plan (see Looking Back).

More will be said later about the evaluations mentioned above; formal means being introduced for making such evaluations. For the moment, however, we shall continue with Polya's views.

Tempo gives an intuitive indication of progress. The pace at which ideas occur to the solver, at which connections are seen and operations are performed, with acceleration sometimes occurring as the nearness of the goal is perceived, suggest confidence in and satisfaction with the current approach. As Polya points out, seldom is an approach abandoned when such momentum exists. In contrast, hesitation or pause tend to signal reassessment and/or realignment, if not uncertainty, preludes to abandonment. Questions arise concerning solvers' levels of awareness for these indicators and of the relevance of recent research on meta-cognition.

Looking Back: For some, this phase begins once an " $x = \underline{\quad}$ " state is reached. The logic of the usual equation solving transformations does not, however, lend support to this, even if no errors are made. Unless reversibility has been maintained at each step, only putative values result which need to be checked in the original equation. It would be a mistake, as indicated by Polya in his discussion of the heuristic syllogism, to equate this state with the answer just because it looks like the typical answer. Aside from such misconceptions, having confidence in their work for the type of equation given, skilled solvers might properly take such a state as a mark of achievement.

Verification is certainly an important part of this phase, but relative to developing equation solving skill, it is also important to review the path which led to the solution looking for those guiding clues and signs of accomplishment that might be used to improve a solvers approach for subsequent equations. The usual school mathematics approach produces a sequence of written steps available for review and analysis. Comparisons and contrasts between stages from the initial to the final can result in the abstraction of key features (characteristics, attributes) free from the nuisance of surface details. Skilled solvers use such features in "devising a plan" and choosing strategies. It would seem, therefore, that a natural

distinction between novice and skilled solvers might be in their knowledge of such features and in their use of them to guide and mark their progress. Then too, by virtue of such abstractions, experienced solvers might more readily recognize an intermediate state along the solution path as being familiar, thus linking to already developed subskills.

Additional Comments: One might consider the above a "common sense" description. The "common sense", however, is that of reflective individuals; skilled solvers in their own right, but far more, individuals knowledgeable of and possessing insights into the mental operations used for such tasks. The ideas concerning signs of progress are largely intuitive in nature needing extension, refinement, and/or modification to be useful for research, curriculum development, and instruction in mathematics education. Protocol collection and analysis would be helpful in this along with examination and discussion of these ideas by mathematics educators. It is in this sense that these comments are offered.

EVALUATION FUNCTIONS

In contrast to the intuitive base for the previous section, the ideas in this section might appear somewhat formalistic. Evaluation functions evaluate states in the search space of a problem, uniquely assigning a vector of one or more values to each. They are typically used with hill-climbing or modified hill-climbing search strategies and can also be used to identify and/or generate subgoals. Wickelgren (1974, page 73) gives the four-dimensional example assigning $(9, 7, 5, 15)$ to the equation $9x + 7y = 5z + 15$ and $(1, 0, 0, \_)$ to the goal state of " $x = \_$ ". He further discusses how one might do successive hill-climbing on the third, second, and first dimensions, respectively, producing the subgoals of $(\_, \_, 0, \_)$, $(\_, 0, 0, \_)$, and then the goal. Whether any solver in the school population might discover and use this particular evaluation function and strategy is an open question. Indeed, this section explores the possibility that evaluation function might be constructed and used to model the direction and progress indicators inherent in the performance and thinking of such solvers.

The equations in the following list are representative of those

- | | |
|---|----------------------------------|
| 1. $26x = 13$ | 5. $2(8x-1) = 18 - (10x+7)$ |
| 2. $2x - 1 = 0$ | 6. $6x^2 - 23x + 10 = 0$ |
| 3. $16x - 2 = 11 - 10x$ | 7. $(x+1)^2 - (x-1)^2 = 2$ |
| 4. $(3x - 10)(2x - 1) = 0$ | 8. $x + 7 = 3(2(2x+1)-5(x-2)-9)$ |
| 9. $\frac{8x-1}{3} = 3 - \frac{10x+7}{6}$ | |

found in the high school curriculum, the use of x as the only unknown being an obvious limitation. Other special features are that all of these equations have $x = \frac{1}{2}$ as a root and that some of them can be seen as intermediate states along typical solution paths for others. Such features as fractions, parentheses, unknowns on both sides, multiple occurrences of the unknown, and exponents, features which have been shown to be important considerations in equation solving (Carry, Lewis, and Bernard, 1979), are represented. It is desirable then, in constructing evaluation functions to construct functions which at least apply to the type of equations in this limited domain.

One such function might be the five-dimensional vector-valued function, say f_1 , with ones entered to indicate the presence and zeros the absence of the above listed features, respectively. The corresponding results from applying f_1 to the equations in the list are shown below.

- | | |
|----------------|----------------|
| 1. (0,0,0,0,0) | 5. (0,1,1,1,0) |
| 2. (0,0,0,0,0) | 6. (0,0,0,1,1) |
| 3. (0,0,1,1,0) | 7. (0,1,0,1,1) |
| 4. (0,1,0,1,0) | 8. (0,1,1,1,0) |
| 9. (1,0,1,1,0) | |

Whether or not these evaluations can be collapsed into meaningful single values, this evaluation function might still be used to solve equations in a fashion similar to the Wickelgren example, hill-climbing first on one dimension and then on another.

A single-valued evaluation function, say f_2 , which is easy to apply and which might be worth considering, is that of assigning the number of typing symbols used to type each equation. The evaluations for the

previous list of equations using f_2 are given in the right-hand column of the next table,

Single-valued evaluation functions have the advantage that they enable the rank ordering of the equations for comparison with rankings by human solvers made along selected dimensions. The following table,

Equation Number	Algebra I	Advanced Algebra	Under-graduates	Graduates	f_2
1.	1.78	1.17	1.00	1.00	6
2.	1.70	2.06	2.00	2.00	6
3.	4.00	4.00	3.14	3.27	12
4.	4.26	4.94	5.86	4.87	15
5.	6.41	5.38	4.57	4.80	18
6.	5.46	5.72	6.71	6.67	12
7.	5.26	6.89	8.14	7.87	15
8.	7.96	6.94	7.29	7.27	23
9.	8.22	8.06	6.29	7.00	27
Group sizes	23	18	14	15	

for example, shows average rankings by certain groups in order of judged difficulty from easiest (1) to hardest (9); along with corresponding f_2 evaluations. It is interesting to note that the correlations between the f_2 evaluations and average rankings from the Algebra I, Advanced Algebra, Undergraduate, and Graduate groups are .96, .90, .69, and .75, respectively. One might be inclined to investigate the significance, if any, of this finding.

Final Comments:

Space limitations do not allow for extensions at this time; hopefully, such will be stimulated by way of the conference discussion. Whether in connection with judged difficulty or some other dimension, it may be that, through creative efforts, evaluation functions will be found to measure solvers' progress from both an external observer's view and also that of the individual solver. Those of the latter type might then be considered for possible inclusion in the curriculum as learning objectives for improving the acquisition and execution of equation solving skills.

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THE PROBLEM OF CONCATENATION

IN EARLY ALGEBRA

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L'algèbre du secondaire est souvent perçue comme une sorte de généralisation de l'arithmétique; et souvent, les enseignants tenteront de corriger certaines fausses conceptions algébriques en utilisant des exemples arithmétiques. Cependant certaines notions algébriques, tel la concaténation (c'est-à-dire la juxtaposition d'une lettre et d'un chiffre indiquant une multiplication), peuvent sembler être comparables à des concepts arithmétiques alors qu'elles en diffèrent fondamentalement. Les résultats de recherches rapportés ici montrent qu'une interprétation arithmétique amène les étudiants à percevoir "3a" en tant que valeur positionnelle, c'est-à-dire 31 ou 32. Une autre interprétation courante que nous avons constatée est celle de subdivision, "3a" indiquant "troisième question, première partie".

As a part of the introduction to algebra the beginning student must learn to understand and manipulate literal symbols and numbers. Included in this introduction is a new concept, that of concatenation - the juxtaposition of a letter and a number or two letters to indicate multiplication, as in $4c$ and xy .

Marilyn Matz in her working paper "Towards a Process Model for High School Algebra Errors" (1979) has suggested that since in arithmetic concatenation denotes implicit addition (for example, place-value, 23, and mixed-fractions, $4\frac{1}{3}$), the novice may fall back on his arithmetic base and consequently conclude that " $4x = 46$, given $x = 6$, or that $xy = -8$, given that $x = -3$ and $y = -5$ ".

The interpretation the beginning algebra student assigns to concatenation may also reflect his understanding of the literal symbol. Kuchemann (1981) refers to two possible interpretations the pupil may give to the letter within the context of an expression such as $3a$, that of "letter as object" (3 a's) and that of first letter abbreviation (3 apples).

Wagner (1981) as well has dealt with students' understanding of variables and has pointed out two common misconceptions: (a) that changing a variable symbol may be viewed by the student as a change in referent and (b) that some students assign numerical values to letters according to their rank in the alphabet. Consequently the meaning that beginning students assign to concatenation might also reflect these two misconceptions.

In our experiment (Constructing Meaning for Algebraic Expressions), as a part of the pretest, we thus included questions aimed at verifying whether or not the problem of concatenation is always present for the novice. The experiment included six students of high, average, and low mathematical ability (three in Grade 6, and three in Grade 7) who had had no formal instruction in algebra. Therefore their responses reflect spontaneous interpretations of concatenation rather than learnt material.

The questions were as follows:

Q.1. When I show you something like this 3a (a card with the expression 3a written on it) can you tell me what does it mean to you?

Q.2. Can you give an example? Can you show me?

Based on Kuchemann's study, the expected responses were that the student might interpret the letter as an object (3 a's), or as a first letter abbreviation (three apples), or as a number, or the intended meaning (3 times the number "a"). The interpretations of the six students were as follows:

- Frankie interpreted 3a as a subdivision label: "third problem, first part".
- Wendy thought of 3a as 31, a place-value interpretation of concatenation, "three one because "a" is the first letter of the alphabet".
- Antoinetta first responded with a subdivision label interpretation ("Number 3, a is the first one), and later with a first letter abbreviation (3 ants).
- Yvette interpreted 3a as a subdivision label ("like 1, a,b,c...") and also in terms of place value (90
-60
3A)
- Filippo assigned to 3a a subdivision label (3A, 3B).
- Gail spontaneously assigned three meanings, first that of subdivision label (3a,3b,...), second that of a first letter abbreviation (a can stand for annex) and third that of place-value (30).

These responses are very different from the ones obtained by Kuchemann. Five out of six students interpreted 3a in terms of a subdivision label, an interpretation which never came to light in the British study. Another unexpected response provided by three students (Wendy, Yvette and Gail) was a place-value assignment to the letter a. Two of the 6 students showed evidence of a first letter abbreviation interpretation.

Q.3. If I show you something like this 3a, does it mean the same thing to you? Can you give me an example? Can you show me?

This question was raised in order to verify if the letter n, as opposed to a, would induce the students to think of n as a number

(based on a first letter abbreviation) and thus result in a subsequent change in their interpretation of concatenation. Another reason for the change of letter stems from Wagner's (1981) discovery of some students assigning a numerical value to a letter according to their rank in the alphabet (as for instance Wendy above).

-Frankie did not perceive $3n$ as being the same as $3a$ and stated that $3n$ is "three times a number or three times n ".

-Wendy thought that the expressions $3a$ and $3n$ were analogous in the sense that the letters could be replaced by their rank in the alphabet ("yes, because n can stand for...whatever number it is in the alphabet") but the two expressions were not the same since the value assigned to the letters were different. Whereas for Wendy $3a$ was 31, for $3n$, "Let's just say it was 13 or 14, I'm not sure, so 313 or 314"

-Antoinetta could not at first assign any meaning to $3n$ since it did not fit into the subdivision label associated with $3a$ because to her, a classification "It usually doesn't go all the way to n ". However, later on going back to viewing $3a$ as "3 ants" she continued this first letter abbreviation and explained that the n in $3n$ could be "a number, a missing number or something".

-Yvette felt that the change of letter did not make any difference. She maintained the two interpretations she assigned to $3a$: thus in the context of a subdivision label, $3n$ was "like $3a, b, c$, up till n "; in the context of place-value she wrote: 681

$$\begin{array}{r} -652 \\ 3n \end{array}$$

-Filippo continued initially with his subdivision label meaning of $3n$, "In question 3 and it goes to $a, b, c, \dots$ and so on till n ". However, the letter n triggered another interpretation as indicated by his statement "Sometimes I guess like 3 times something equals 12 or 3 and 4..." which he wrote as " $3n4 = 12$ ". Filippo thought some letters such as n or h (but not a) could be used "like a blank" to express some unknown operation as in the example above "but x they wouldn't use, because it would mean multiplication".

-Gail did not think that $3n$ was the same as $3a$ since in the subdivision label interpretation "in elementary we don't have n 's, we only have $3a$ or $3b$ ". When asked to assign meaning to $3n$, she reverted to her first letter abbreviation interpretation and wrote "3 numerators" for which she gave as example $\frac{n}{n} + \frac{n}{n} + \frac{n}{n}$ for which she later

assigned different values $\frac{\frac{2}{n} + \frac{1}{n} + \frac{1}{n}}{12} = \frac{12}{12}$.

These responses bring out the strength of the subdivision label interpretation. Of the five students who initially used this interpretation for $3a$, three of them maintained it for $3n$, while for the other two (Antoinetta and Gail) this initial perception created a

conflict. Two of the three students who had initially indicated a place-value interpretation with 3a gave further evidence of this with 3n (Yvette and Wendy).

The change in letter from a to n appears to have resulted in some unexpected changes in interpretation of the letter in the context of concatenation. Only in the cases of Frankie, who used the intended meaning for 3n (3 times a number), and of Antoinetta, who referred to n as the missing number, was there any evidence that students spontaneously think of n as opposed to a as a number based on first letter abbreviation. The n in 3n resulted in an entirely different interpretation for Filippo, that of a 'missing operation', and for Gail thinking in terms of first letter abbreviation did not lead her to interpret 3n as 3 numbers, but rather 3 numerators.

Wendy's response reinforces Wagner's discovery. To Wendy 3a is 31 and 3n is 313 or 314. It is interesting to note that for Wendy the letter in 3a and 3n, is not interpreted just as place-values, but rather as placeholders for numbers in some place-value system. The value of the letter is always determined by its position in the alphabet.

Of the six students, only Yvette thought that the expressions 3a and 3n had the same meaning; to Wendy they were similar in the sense that the letters represented their position in the alphabet, but for the other four students the two expressions could not be accepted as equivalent.

Q.4. If I ask you to replace here 3a the letter a by the number 2, can you tell me what you get?

This question was aimed at determining whether or not those students using a first letter abbreviation (a for apples) or a letter-as-object (3 a's) interpretations would in fact use multiplication upon being asked to substitute.

-Frankie at first responded "I would put 3 times or plus by 2" and wrote 3×2 . But substitution brought other interpretations: "or there is another thing (he wrote 3²), the base 3" and "or also,

sometimes there is a number on the bottom, it would show you (he wrote 3_2)". Identical interpretations were obtained by substituting 2 in $3n$.

-Wendy responded "O.K. thirty-three, because the a is one adding on the two would be three, or if you just drop the a and put 2 it would be 32". When asked to substitute 2 in $3n$ she wrote "32 or 17" the latter answer being justified by "I just counted n is 14...14 plus 3...". However, she later reverted to the initial procedure she used for substituting in $3a$, "because n is 14 plus 2 is 16, that would be 316".

-Antoinetta departed from her initial interpretations of concatenation and upon substituting 2 in either $3a$ or $3n$ responded with "32" thereby indicating a place-value interpretation.

-Filippo also diverted from his initial interpretations of concatenation and responded with 32, a place-value meaning.

-Gail upon being asked to substitute responded with "thirty-two or three times two or three plus two...or three minus two" and wrote $3 - 2$, 3×2 , $3 + 2$, 32.

The substitution question proved to be most interesting in terms of unexpected responses. All students, except Frankie, used a place-value interpretation. Although Wendy, Yvette and Gail had given some prior evidence of this, Antoinetta and Filippo had not. The two students (Antoinetta and Gail) who had indicated that they could use a first letter abbreviation did not continue this when they substituted although Gail as well as Frankie did mention "3 times 2" along with other possible operations.

The following table provides a summary of each student's responses to the three concatenation questions.

	Frankie	Wendy	Antoinetta	Yvette	Filippo	Gail
	3a 3n sub	3a 3n sub	3a 3n sub	3a 3n sub	3a 3n sub	3a 3n sub
Subdivision Label	✓		✓✓	✓✓	✓✓	✓✓
Place-Value		✓✓✓	✓	✓✓✓	✓	✓✓
1st Letter Abbre.	✓			✓✓		✓✓
Multiplication	✓✓					✓
Other	✓				✓	✓

TABLE I - Students' responses to the three concatenation questions

(sub = substitution)

SUMMARY

What is striking in these results is that the two most prevalent interpretations chosen by five out of the six students (subdivision label and place-value) did not appear at all in Kuchemann's study and the place-value interpretation was merely hinted at by Matz. The strength of these interpretations is evidenced by their consistent use: the subdivision label was used twice by four of the students, whereas the place-value interpretation was used three times by two students and twice by another.

While none of our subjects used letters as objects in this context, three students did indicate a use of literal symbol as a first letter abbreviation. Two of these three students did in fact indicate that 3a meant multiplication when a was replaced by 2. However, this substitution task brought the place-value interpretation among two other students who previously had not made any reference to it.

Interestingly enough the change of letter from 3a to 3n only induced one student (Frankie) to think of multiplication while

another one (Antoinetta) did mention "number" without any reference to multiplication and, a third one (Gail) looked for a word starting with the letter n and found "numerator". Thus the change of ~~the~~ did not provide the expected evidence of change to a numerical referent in the students.

Almost all the students (5 out of 6) did not perceive 3a as being the same as 3n. Antoinetta and Gail explained it in terms of classification no usually extended to n parts; Frankie shifted from subdivision label to multiplication; Filippo shifted from subdivision label to an arithmetic operation; Wendy perceived them as different since to her the letter represented a number determined by its position in the alphabet.

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RELATIONSHIPS BETWEEN NOVICES' VIEWS OF ALGEBRAIC LETTERS AND THEIR
USE OF SYMMETRIC AND ASYMMETRIC EQUATION-SOLVING PROCEDURES

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RESUME

Une étude en trois phases a été menée dans le courant de l'année 1980-1981. La première phase consistait en entrevues avec 10 étudiants de 7^e année (agés entre 12 ans et demie et 13 ans et demie) afin de découvrir quelques-unes de leurs notions pré-algébriques. Un sous-groupe de 6 étudiants a été retenu pour la seconde phase qui consistait en une expérimentation didactique d'une durée de trois mois sur la résolution d'équations algébriques. La troisième phase impliquait des entrevues avec 9 sujets (de la 8^e à la 11^e année) plus expérimentés avec l'algèbre pour investiguer leur façon de résoudre des équations, leurs erreurs, et leurs méthodes de résolution dans la détermination de l'équivalence de paires d'équations. Ces méthodes plus expérimentées ont été comparées avec les méthodes employées par les débutants. L'objet de cet article consiste principalement à présenter quelques résultats issus des deux premières phases. Ceux-ci indiquent une forte prédominance à l'utilisation de la transposition et de l'opération inverse comme méthode de résolution d'équations de la part des sujets plus expérimentés. Par ailleurs, les sujets non-expérimentés avaient tendance à se situer dans deux catégories distinctes caractérisées par une approche asymétrique (utilisant la transposition) ou une approche symétrique (utilisant leurs connaissances des nombres, les substitutions, ou l'exécution de la même opération des deux côtés). Leur préférence pour une approche plutôt qu'une autre était reliée à leur vision des lettres dans les équations et les expressions algébriques. Ceux qui voyaient les lettres comme la représentation d'un nombre dans une équation algébrique donnée avaient tendance à préférer les procédures de résolution symétriques. Par contre, ceux qui n'accordaient aucune signification à la lettre contenue dans une équation donnée et la considéraient comme un simple résultat d'un ensemble d'opérations inverses avaient tendance à préférer les procédures asymétriques.

Algebra is known to be a school subject which presents many cognitive obstacles to the student encountering it for the first time (eg, Booth, 1981; Kieran, 1979; Matz, 1979). Some studies (Kuchemann, 1978; Wagner, 1981) have shown that novices interpret the algebraic letter in a variety of underdeveloped and sometimes incorrect ways; other studies (Jensen, Rachlin, & Wagner, 1982) have indicated the kinds of errors that students tend to make when solving equations and simplifying expressions; and yet others (Petitto, 1979) have investigated some of the approaches used in the process of solving equations. However, there is still very little known on how these different aspects of algebraic understanding and performance interact. For example, what kinds of links are there between a novice's view of letters in equations and the methods he uses to solve equations. It is the purpose of this paper to elucidate some of these links.

A three-phase study was carried out during the school year 1980-1981. The first phase involved interviews with 10 seventh graders, with a view to ascertaining some of their pre-algebraic notions (Kieran, 1981b). A subset of this group ($n = 6$) was retained for the second phase of the study: a three-month teaching experiment on equation-solving. Topics covered ranged from constructing meaning for the concept of equation to solving a variety of equations with several occurrences of the unknown. The solving method emphasized was that of performing the same operation on both sides. The third phase of the study involved interviews with 9 more-expert algebra students (they had all taken at least one algebra course) from grades 8 - 11 to uncover their equation-solving procedures, errors, and methods of determining the equivalence of equation-pairs. These more-expert methods and errors were compared with the corresponding novice methods and errors.

This paper will focus on some of the results of the first and second phases of the study -- those results which deal with the processes which algebra novices use initially in the solving of certain types of equations and the ways in which these processes interact with their views of the letter in an algebraic equation.

METHOD

First Phase:

Subjects

The subjects were 10 students in the seventh grade ($12\frac{1}{2}$ - $13\frac{1}{2}$ years), all of average mathematical ability. They had not yet taken algebra.

Procedure

One 50-minute interview was held with each subject. The interview had as its purpose to ascertain the kinds of pre-algebraic knowledge these subjects possessed. A variety of questions were asked, including some which touched upon the subject's views of the letter in algebraic equations and expressions, such as, a) What does the letter mean in i) $5 + \underline{a} = 12$, ii) $\underline{n} \times 3 = 15$, iii) $9 \times 4 = \underline{b}$, iv) $2\underline{c} + 15 = 29$, b) What does $\underline{a} + 3$ mean?, c) What does $3\underline{a}$ mean?

Second Phase:

Subjects

The subjects were 6 students in the seventh grade (4 boys and 2 girls). This group was a subset of the group of 10 first-phase subjects.

Procedure

For each subject, there was a pretest interview, eight to ten teaching interviews, a posttest interview in June at the end of the teaching experiment, and a second posttest interview the following September after the summer holidays. Each interview was audio-taped and transcribed in its entirety.

The pretest and both posttests included a set of 14 equations to be solved. Each teaching experiment interview was usually begun and ended by having the subject solve a set of 12 equations. Each set contained equations of the following types: $6\underline{b} = 24$, $32\underline{a} = 928$, $\underline{n} + 6 = 18$, $\underline{y} + 596 = 1282$, $4\underline{c} + 3 = 11$, $13\underline{x} + 196 = 391$, $2\underline{x} - 6 = 4$, $16\underline{x} - 215 = 265$, $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$, $3\underline{a} + 5 + 4\underline{a} = 19$, $2 \times \underline{c} + 5 = \underline{c} + 8$, $4\underline{x} + 9 = 7\underline{x}$.

Most of the data for this paper are taken from the parts of the second phase protocols which had an equation-solving component. The remaining data are from the first phase of the study, the phase which

included interview questions on the subjects' views of the letter in algebraic equations and expressions. In addition to subjects' responses to these specific questions in the first phase, we have also included in our analysis their unsolicited comments on letters which they made while attempting to solve the second-phase equations.

RESULTS AND DISCUSSION

The results of the pretest indicated that the 6 novices of this study solved 1st degree algebraic equations in one unknown by three methods:

- i) known number facts (eg, in $6b = 24$, subject replied "6 times 4 equals 24");
- ii) substitution (eg, in $2x - 6 = 4$, subject replied "2 times 6, 12, minus 6, 6, no, 2 times 5, 10, minus 6, 4, yes");
- iii) inversing (eg, in $6b = 24$, subject replied "24 divided by 6, 4").

Subjects solved the 14 equations of the pretest by using known number facts 24 times, substitution 26 times, and inversing 49 times. Thus, known number facts and substitution were together used as often as inversing; however, three subjects used the first two methods more often than the last one, and the other three subjects used inversing more often. A similar pattern was seen in the June posttest when subjects used known number facts 18 times, substitution 17 times, inversing 77 times, and the procedure of performing the same operation on both sides 28 times.

Pretest data also indicated that even though all six subjects used known number facts in varying degrees, two never used substitution, preferring instead the inversing method. Nevertheless, it seemed clear that at least half of the novices were not restricting themselves primarily to the inversing method. This is in contrast with the more-expert subjects of the third phase of the study who all strongly preferred the inversing method. The nine more-expert subjects of the study used inversing 152 times, known number facts 9 times, substitution once, and the procedure of performing the same operation on both sides 9 times.

The more-expert subjects' heavy usage of the inversing method suggested that this method might be conceptually quite different from the other solving methods. Therefore, we decided to take another look at the data to try to uncover what these differences might be, and to find out to what extent some novices seemed to prefer inversing over the other methods. What we discovered was that the equation-solving methods of the novices of our study tended to fall into two fairly distinct categories, symmetric vs. asymmetric approaches, and that the preference for one approach over the other was related to their view of the letter in algebraic equations and expressions.

Asymmetric vs. Symmetric Solving Methods

For simple one- and two-operation equations with the unknown on the left, the procedure of inversing involves working backwards, from the side without the unknown, usually from right to left. The resulting equation has a very asymmetric look to it, for all of the operations appear on one side (eg, $4x + 5 = 9$ ends up as $x = (9 - 5) \div 4$). Even more complex equations which are solved by this inversing method maintain an asymmetric look to them as terms are systematically deleted from one side and placed on the other.

On the other hand, the use of known number facts, substitution, and the procedure of performing the same operation on both sides does not entail any asymmetric alteration of the given structure of the equation. Solutions are found by using the given numbers and operations, generally in the given order, subject to the constraints of the conventional order of operations. In contrast to the asymmetric solution path obtained when one solves, for example, $4x + 5 = 9$, by the inversing method, the method of performing the same operation on both sides produces a very symmetric solution path:

$$\begin{array}{rcl} 4x + 5 & = & 9 \\ 4x + 5 - 5 & = & 9 - 5 \\ 4x & = & 4 \\ 4x \div 4 & = & 4 \div 4 \\ x & = & 1 \end{array}$$

Thus, there is a very clear difference at the surface level between the inversing method, which we are calling an asymmetric method, and

the other three solving methods, which we are calling symmetric. However, another difference exists at a deeper level. It involves a complex interaction with the way novices view letters in equations.

View of the Letter in Algebraic Equations

Among the 6 algebra novices, there were two distinct ways of looking at the letter in equations: one view considered letters as representing numbers within the given equality relationship; the other view ascribed no meaning to the letter within the given equation, but rather saw the letter as the result of performing a set of inverse solving operations.

According to the latter view, meaning for a letter is derived from the operations one performs to find its value; the letter in fact only has meaning once its value is found. In contrast, those who hold the former view see the letter as something in its own right in the given equation -- as a number contributing to the numerical equilibrium of the equation, and not just as the result of a set of solving operations. The inverse-operation view of the letter would seem to be consistent with the asymmetric method of equation-solving and could be called an asymmetric view of the letter; whereas, the other view would be consistent with symmetric solving methods and could be called a symmetric view of the letter.

View of the Letter in Algebraic Expressions

These two ways of viewing the letter in an equation can also account for two different types of responses given in the algebraic expression tasks of the first phase of the study (Kieran, 1981b). Subjects were asked what the letter meant in expressions, such as, $a + 3$. Subjects who were strongest in terms of defining the letter of an equation as the result of a set of inverse operations were unable to assign any meaning to $a + 3$. Since it lacked an equal sign and right hand member, a had no meaning. On the other hand, subjects who were less inclined to define the letter of an equation as the result of a set of computations were able to view the a in $a + 3$ as being some number or as any number.

Comparison between View of Letter and Preferred Solving Method

Three of the six novices had a view of the letter consistent with symmetric procedures of equation-solving. Two of these consequently preferred to use symmetric procedures when solving (in both pretest and posttest). On the other hand, two subjects had a view of the letter which was consistent with asymmetric solving procedures. One of these two never used substitution in solving the pretest equations, and thus clearly preferred asymmetric solving procedures; whereas, the other one did not show a clear preference for asymmetric solving procedures. We shall return to this subject later. The sixth subject of the study seemed to hold both views of the letter, and used both methods just about equally when solving equations. Thus, for 4 of the 6 novices, we have a match between their views of algebraic letters and their preferred methods of equation solving in the pretest.

One of the two remaining subjects, in which this match did not occur, had a view of the letter which would have favored symmetric solving procedures. Yet the method he used in most of the pretest was an asymmetric one. What is interesting about this subject, however, is that by the time of the June posttest, he had substantially increased his usage of symmetric procedures, thereby coming closer to matching with his view of algebraic letters. The other remaining subject, in which the pretest match did not occur, had a view of the letter which favored asymmetric procedures. Though the solving methods she used in the pretest were primarily symmetric, she steadily dropped these methods throughout the teaching experiment in favor of asymmetric ones. She did not seem to feel at all comfortable with the symmetric approaches being used during the experiment. Thus, by the time of the June posttest, her preferred equation-solving method was clearly asymmetric.

This points out that, for both of the subjects in which a match did not occur in the pretest between their view of the letter and their preferred method of equation-solving, a match did occur, by and large,

in the June posttest. For the four subjects in which a match occurred in the pretest, the same match occurred in the June posttest.

All of this suggests something important with respect to the instruction that took place between pretest and posttest. The instruction of the teaching experiment was very strongly oriented toward the symmetric procedure of performing the same operation on both sides. The evidence indicates that it had its greatest effect on those novices who already had a view of the letter which was consistent with symmetric procedures. On the other hand, it seemed to have very little effect on those who had what we call an asymmetric view of the letter. This seems to underline the point that algebra novices have a certain predisposition toward symmetric or asymmetric solving procedures based on their elementary school experience with letters, and that algebra instruction in secondary school which favors this predisposition will be successful; whereas, instruction which does not, will be less successful.

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ISSUES IN COGNITIVE EXPLANATIONS OF ALGEBRAIC MANIPULATIVE SKILL
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RESUME

Les explications cognitives des habiletés mathématiques deviennent de plus en plus populaires. Les techniques de simulation par ordinateur constituent une très bonne méthode pour tester et élaborer ces théories mais il y a, dans la simulation d'écriture, une forte tendance à se concentrer sur des problèmes minimes d'exécution au lieu de s'attaquer aux questions psychologiques de plus grande importance. L'auteur de ce travail est engagé dans une analyse linguistique qui permettra, nous l'espérons, la confrontation avec d'importantes questions sur la manipulation de symboles algébriques. Le modèle linguistique peut être comparé aux méthodes de recherche sur le traitement de l'information selon trois différentes perspectives: la nature des théories; la généralisation des résultats; les méthodes pour confronter les diverses explications cognitives. Pour terminer, un exemple du travail de l'auteur sera brièvement présenté pour illustrer les relations possibles de différentes théories avec la syntaxe algébrique et pour proposer un modèle de recherche psycholinguistique pouvant être utilisé dans le choix d'une alternative.

PRELIMINARIES

The study of cognitive processes generally and of mathematical skills in particular increasingly utilizes information processing (IP) and artificial intelligence (AI) terms and methods. The character of much of this research is well expressed in the self description of The Journal of Mathematical Behavior to be found on page vi of recent issues.

- 1) Data is often obtained from task based interviews, somewhat in the style of Piaget and Krutetskii;
- 2) Attention is paid to thought processes, to the question of how the student was thinking about some mathematical task;
- 3) Behavior is typically analyzed in relation to a postulated theory of human information processing;
- 4) The basic conceptualization of human information processing tends to be drawn from the field of cognitive science or artificial intelligence, as in the work of Newell, Simon, Minsky, Papert, Winograd, Winston, Matz, Rissland, and John Seely Brown.

Generally, research in the area reports on students' errors or on an interaction involving students (less often involving skillful mathematicians) and then seeks to explain the observations in information processing terminology. The mathematics investigated ranges through the K-12 curriculum with a sampling of university material and covers the meaningful and the rote, the word problem and the manipulation of symbols, the novel and the familiar, the successful and the unsuccessful.

Certainly one of the major attractions of this research is its attempt to describe diverse aspects of mathematical performance. There are many educators, both researchers and practitioners, who have a great deal of insight into some aspect of mathematical skill and its development. It is precisely the absence of a uniform conceptual and terminological framework that isolates these insights one from another and impedes their incorporation into a truly

effective curriculum.

Whether or not this ambitious program will be realized in the present terms and concepts of information processing theory, or indeed at all, is speculation best left to hindsight. Certainly, as the behaviorist program indicate, the talent, energy and enthusiasm of the proponents is no guarantee of success. IP theory, however, has an advantage over previous attempts. The computer provides for the comprehensive testing of the computational theory presented as artificial intelligence program. "Proof" of the adequacy or inadequacy of a theory is a welcome alternative to speculation.

Computer testability, while an advantage, is only a limited advantage. A computational theory, to be tested, must be fully operationalized with the result, as Davis and McKnight (1979) note, that

the general tendency of simulation-writing is to be pushed toward dealing with minute details - and frequently details that are more characteristic of computers than they are of human thought.
(p. 31)

Shank (1977) expresses the need for insight.

Researchers are starting to understand that tour-de-force in programming are interesting but non-extendable . . . the AI people recognize that how people use and represent knowledge is the key issue in the field. (p. 1007-1008)

And David Marr (1981) suggests that other modes of structured inquiry can provide the needed insights:

Chomsky's (1965) notion of a "competence" theory for English syntax is precisely what I mean by a computational theory for that problem. Both have the quality of being little concerned with the gory details of algorithms that must be run to express the competence (i.e., to implement the computation). That is not to say that devising suitable algorithms will be easy, but it is to say that before one can devise them, one has to

know what exactly they are supposed to be doing, and this information is captured by the computational theory. (p. 131)

Is it possible to adapt Chomsky's linguistic methods to the study of mathematical skill in order to obtain a principled account of the latter? The research in which I am presently engaged is an adaptation of modern linguistic and psycholinguistic techniques to the domain of algebraic symbol manipulation. These methods have some distinct advantages and disadvantages to IP methods which I would like to consider today.

THE NATURE OF THE FORMALISM

Psychological phenomena are represented in IP studies in terms of the basic constructs of the theory (goal stacks, VMS sequences, etc.). Ultimately, the goal is to rigorously represent the phenomena of interest in terms of these constructs so that all of the advantages of a formal theory may accrue to the study of cognition.

The basic constructs, however, are by no means independent of the phenomena under investigation. Indeed they are themselves perceived to be "metaphors" for cognitive structures and processes:

We . . . focus . . . on another key aspect of recent studies: the interpretation of observed behavior in terms of metaphoric models of human information processing - the analogue of the physics/chemistry metaphoric models of atoms, molecules, and other particles or aggregates of particles.

(Davis and McKnight, 1979, p.95)

Is the system of metaphors currently employed sufficiently powerful, to adequately represent the phenomena of interest? One way to attempt to answer that question is to formalize IP theories to the extent possible and carefully evaluate the results. This proposal must be delicately undertaken, however, since premature formalism will surely curtail the

very insights needed to strengthen the metaphoric base. In short, the program of formalism in IP cannot now, nor possibly for a considerable time to come, make unconstrained use of the formalism to which it seems so well suited.

Linguistics, by contrast, has its formalism rooted in its conception of "language" rather than mind. The concern to maintain "psychic distance" from the object of study is of primary importance to the paradigm. Nevertheless linguistics does make cognitive claims:

. . . a generative grammar is not a model for a speaker or a hearer. It attempts to characterize in the most neutral possible terms the knowledge of the language that provides the basis for actual use of language by a speaker-hearer. (Chomsky, 1965, p. 9)

Linguists are thus in a position to enjoy the advantages of a fully formal methodology.

Unfortunately, concomitant with the strength of the formalism is a restriction on the kinds of entities to which linguistic methods can be applied. Chomsky (1957) defines a "language" for the purposes of linguistic analysis.

"From now on I will consider a language to be a set (finite or infinite) of sentences, each finite in length and constructed out of a finite set of elements. All natural languages in their spoken or written form are languages in this sense, since each natural language has a finite number of phonemes . . . Similarly, the set of 'sentences' of some formalized system of mathematics can be considered a language."

Thus the linguistic paradigm is useful for the study of the manipulation of algebraic symbols, but cannot be extended to broader aspects of word problem or novel problem solution.

GENERALIZABILITY OF FINDINGS

The attempt in IP research to deal both with algebraic competence and its development in a single theory, presents

some difficulties. The modeling of developmental processes has inherent in it certain obvious but inescapable problems. The most prominent of these is the instability of the data available to the observer. By their nature, developmental processes are difficult to capture initially, and more difficult still to replicate. The response which is present today may be gone tomorrow. For this reason it is not surprising that the dominant paradigms for the collection of data are the case study and clinical interview. As Matz (1980) righteously declares in her footnoted comment of Sussman's statement:

"Statistics is the science of populations, artificial intelligence is the science of the individual." (p. 97)

It should be recognized, however, that the instability of acquisition data and the attendant problems of the generalizability of findings is a serious limitation of the paradigm which will doubtlessly impede the acceptance of its results by a generally conservative educational establishment.

In linguistics there is a sharp delineation between the description of competence, and the description of its acquisition. The subject matter of each is of great consequence to the other, however, the methodological constraints are distinct. The "informant" who provides the corpus of data or the "judgements of grammaticality" for the grammarian is generally conceived of as a representative of a homogeneous speech community. The status of the data which the developmental linguist obtains is far more problematic. The dialectical relationship between acquisition and competence grammars is of interest, but time does not permit its discussion today.

LIMITS OF EXPLANATION

A mark of a mature science is the clear delineation of the

bounds of explanation available to it. If IP terminology provides for descriptions of cognitive events, and AI programs assess the internal consistency of these theories, then how is the validity of the theories to be determined? Simon and Newell (1971) recognize the problem:

Programs like LT [The Logic Theorist] are explanations of human problem-solving behaviour only to the extent that the processes they use to discover solutions are the same as the human processes. (p. 147)

There seems to be some suggestion in the literature (Paige and Simon, 1966) that AI techniques can be used to resolve validity issues as well. Basically, the method employed involves specifying to ever greater degrees the requirements which the simulation must meet.

Hence, the theory may be regarded as a first approximation that explains some of the broad outlines of the human behavior, and, by allowing us to abstract from these, brings into sharp focus additional phenomena that might otherwise escape our gaze. A next task . . . is to enlarge the theory into a second approximation so that still finer detail of the behavior can then be detected as deviations from this more accurate model.

The procedure just outlined has become a standard for testing information processing theories. To be worthy of consideration, such a theory must pass the test of sufficiency. It must be able to handle cognitive tasks comparable to those handled by human subjects at some level of skill. If it passes the sufficiency test, it is tested in more detail by comparing its behavior with human behavior on sets of identical or similar tasks. (p. 55-56)

There is no a priori justification for the belief that only a single IP structure will be able to account for human performance. Indeed, it will be a tribute to the strength, not the weakness of the paradigm if alternative conceptions of mind are equally well represented in its terminology. To the extent that this occurs, it will be necessary to devise means for selection on psychological grounds.

Linguists have also skirted the issue to some degree. Chomsky's (1965) solution to the selection problem is to choose the grammar which is most easily related to a general theory of language structure (a universal grammar).

Regardless of the merits of this position (and it is controversial in linguistic circles), no entity corresponding to "universal grammar" is postulated for algebra. It is therefore fortunate that psycholinguistics, a field which has developed in parallel with linguistics, offers a range of fully articulated methods to assess the relative psychological merits of alternative grammars.

In developing the grammar of algebraic symbol manipulation, numerous alternative components have been postulated which equally well fulfill the formal requirements of the grammar. I will conclude my talk today with one example of alternative components to the grammar and illustrate the psycholinguistic experiment which was used to make a selection. This example, regarding the "syntax" of algebra, will illustrate the way in which such questions are formulated, and present a typical psycholinguistic experiment. The question selected is, I think, also of interest on its own merits, demonstrating the strength of algebraic notation, and the initiative and independence which must be attributed to the successful student.

ALGEBRAIC SYNTAX

In an algebraic expression, a definite structure associates the component numerical and variable ("quantity") symbols. For example in $3x^2$ it is the x which is squared and the result which is tripled, not the x which is tripled and the result which is squared.

A concise description of this "syntax" utilizes a taxonomy of operational levels introduced by Schwartzman, (1977):

Taxonomy of Operation Levels

level 1: $+$
level 2: multiplication, division
level 3: exponentiation, $\sqrt{\quad}$

(In this hierarchy, level 3 is said to be higher than level 2, which is higher than level 1.)

The following rule assigns syntactic structure to expressions. It corresponds closely to standard pedagogical devices such as BOMDAS which is an acronym for Brackets, Of, Multiply, Divide, Add, Subtract, (except that, mysteriously, BOMDAS assigns no precedence to the exponentiation and radical operations).

Syntactic Rule: Operator precedence is assigned to the operation contained within the greatest number of parenthesis pairs. In case of a tie:
1. The highest level of the competing operations receives precedence. In case of a tie:
2. The left-hand most operation receives precedence.

An interesting feature of our algebraic notation is that it is possible to specify operation level without explicit reference to operations by utilizing redundant spatial characteristics of the algebraic symbol system:

Alternative Taxonomy of Operation Levels

Level 1: wide spacing between the quantity symbols ($A \quad B$)
Level 2: vertical or horizontal concatenation (AB , $\frac{A}{B}$)
Level 3: diagonal concatenation (A^B , $\sqrt{B}$).

The consequence of this alternative version is that operation precedence is assigned by the Syntactic Rule with no explicit reference to operations at all.

When alternative components are formulated for a grammar which equally well fulfill its formal requirements, then one may ask which formulation most adequately describes the cognitive functioning of the competent algebraist?

The study in which I addressed this syntax question uses a methodology common in the psycholinguistic literature for the study of redundant features. In this paradigm, a nonce form is contrived which incorporates only one of the redundant features. Subjects are then asked to perform linguistically using the nonce forms. If they are able to function successfully in these circumstances, then this is counted as evidence that the feature incorporated in the nonce form is the psychologically active one. If they are unsuccessful, then the opposite conclusion is reached.

In a study conducted earlier this year, 630 subjects including students in grades 9, 11, 1st year calculus and 4th year engineering as well as twenty professional engineers were asked to evaluate algebraic expressions such as $1A3MxE2$ where A, M, and E stand for addition, multiplication and exponentiation respectively, and "x" has the value 2. Calculations were all done mentally to discourage translating back to standard notation. The nonce form allows the subjects access to the propositional information regarding the operations comprising the algebraic expressions, however it distorts the spatial relations between symbols in the standard notation. All subjects were then asked to perform similar tasks using the standard notation. 506 subjects answered all of the questions and achieved greater than 70% on the regular notation computations.

The grand mean for all of these subjects on the nonce form tasks was 59.2% as compared to a mean of 95.7% on the standard notation.

To guard against the possibility that it was just the unfamiliarity of the notation that interfered with the subjects performance, rather than the specific spatial characteristics as hypothesized, about half of the subjects

had been randomly selected in each of the classes tested to use an alternate form of the nonce notation:

1 A 3 M xE2 instead of 1A3MxE2. For these subjects, the notation used incorporates approximately the same spatial relations as the standard notation. The grand mean for subjects using this notation was 74.3%. This mean is significantly different from 59.2% at the .001 level.

These results indicate that for some people, at least, the rules which they have internalized to make the syntactic decisions which are fundamental to algebraic competence are best described in terms of the positions of symbols as distinct from the propositionally encoded information about operations. Clearly these rules bear little relation to the knowledge of syntax - if any - acquired through textbooks or instruction since the knowledge disseminated through these channels is almost entirely propositional in nature.

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THE VARIABLE-REVERSAL ERROR AMONG FIVE CULTURAL GROUPS

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RESUME

Nous présenterons, ici, les résultats d'un examen sur le raisonnement proportionnel, examen donné à des élèves Fijiens (Hindu et Fijiens), Israéliens, Japonais et Américains dans leur pays d'origine. Les résultats de l'examen indiquent qu'il existe une différence manifeste entre ces groupes. En outre, tous les groupes ont trouvé plus difficile d'écrire une formule reliant des grandeurs interdépendantes lorsque la présentation était picturale que lorsqu'elle était écrite.

The Variable-Reversal Error Among Five Cultural Groups

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Several recent studies (Clement, Lochhead & Monk, 1981; Clement, 1982; Mestre, Gerace & Lochhead, 1982) have revealed the widespread inability of college science and engineering students to translate a simple statement expressing an algebraic relationship between two variables into an equation. Consider the following problem:

Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university". Use S for the number of students and P for the number of professors.

Incorrect answers were given by 37% and 57% of the engineering and non-science students tested, respectively (Clement, et al., 1981). Using a sample of Hispanic engineering majors, Mestre, et al. (1982) found an error rate of 68%. By far, the most common error committed consisted of the "variable-reversal" error, namely $6S=P$.

Clinical interviews of students solving this problem revealed that the variable-reversal error often appears to derive from a sequential left-to-right translation of the problem statement into an algebraic equation. In the "students and professors" problem, it means that "six times as many students" gets mapped to $6S$, and this quantity is equated to P. Students writing $6S=P$ often did not treat S and P as variables to represent the number of students and the number of professors; rather, they treated S and P as labels for the nouns, "students" and

"professors", respectively.

Some of the Hispanic subjects in the Mestre, et al., study exhibited unique errors deriving from misinterpretations of the problem statement. For example, in the "students and professors" problem, some Hispanics gave the answer $6S=6P$. In clinical interviews, these students stated that the phrase "as many students as professors" meant that there were an equal number of each, that is, $S=P$; they explained that the "six times" in front meant that each side of $S=P$ should be multiplied by 6. Another error not found among an Anglo control group was the equation $6S+P=T$. Students who wrote this answer explained that this equation related the number of students, professors, and the total student-professor population, T , in the appropriate proportions. Wollman (1983) also found this response among female elementary education majors. He found that, unless specifically warned that the "students and professors" problem was not asking for a relationship among students, professors, and the total student-professor population, his subjects would sometimes write an equation involving an additional variable representing the total of the two quantities in question.

Another interesting finding (Clement, 1982; Mestre, et al., 1982) concerns the increase in the error rate when students were asked to write an equation from a pictorial presentation of the problem. That is, in a problem such as the "students and professors", subjects are shown a picture of a university scene with 6 students and 1 professor, then told that this picture is representative of the relative population of students and professors, and finally asked to write an equation relating the total number of students and professors at the university.

In this paper, we will present data from a five question test consisting of problems such as the "students and professors", and other ratio-type problems (both direct and inverse proportions). The test was given to students in Fiji, Japan, Israel, and the U.S. We will first describe the test and the subjects taking the test. This will be followed by the results of the test. We will conclude with a brief discussion of the results.

SUBJECTS

Fijians: Two distinct ethno-linguistic groups of Fijians took part in the study. The first group consisted of 40 female and 39 male Hindi-English bilinguals. The second group consisted of 20 female and 18 male Fijian-English bilinguals. Both groups were second year college students enrolled in the Diploma in Education program. English is the language used in the schools in Fiji, and the test was given to the students in English. All students had taken a math course each year in school prior to entering college.

Israeli: The Israeli group consisted of a college and a pre-college group. The college group consisted of 14 female, and 27 male freshman agriculture students who were taking a calculus course. The pre-college group consisted of 60 female and 43 male 7th to 10th graders. Most students were bilingual, with Hebrew being the language understood by all the students. The test was given in Hebrew.

Japanese: The Japanese group consisted of 59 female and 112 male undergraduates enrolled in a social science program (most in sociology or pre-law) in a university in Tokyo. These students had either taken or were currently enrolled in courses in elementary statistics and computer programming. The test was administered in Japanese.

American: The American group consisted of 37 female and 48 male freshman undergraduates enrolled in a math problem-solving course for non-technical majors at the University of Massachusetts. According to a math placement exam, these students were not prepared to enroll in a pre-calculus course.

THE TEST

The five questions in the test were the following:

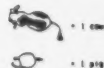
1. Write an equation using the variables X and Y to represent the following statement: "In one school, there are six times as many students as there are teachers". Use X for the number of students and Y for the number of teachers.

2. A man owns 2 cars and drives one of them to work every day. His first car gets 12 miles per gallon and uses 3 gallons for this trip. His second car gets 18 miles per gallon and uses ? gallons for this trip.

3. A man takes a photograph from an airplane of some of the cows and pigs in a large field full of cows and pigs. He is sure that he has photographed a typical sample of the animals in the field.

Write an equation using the letters C and P to describe the relationship between C , the number of cows, and P , the number of pigs in the field. The equation should

allow you to calculate the number of cows if given the number of pigs.



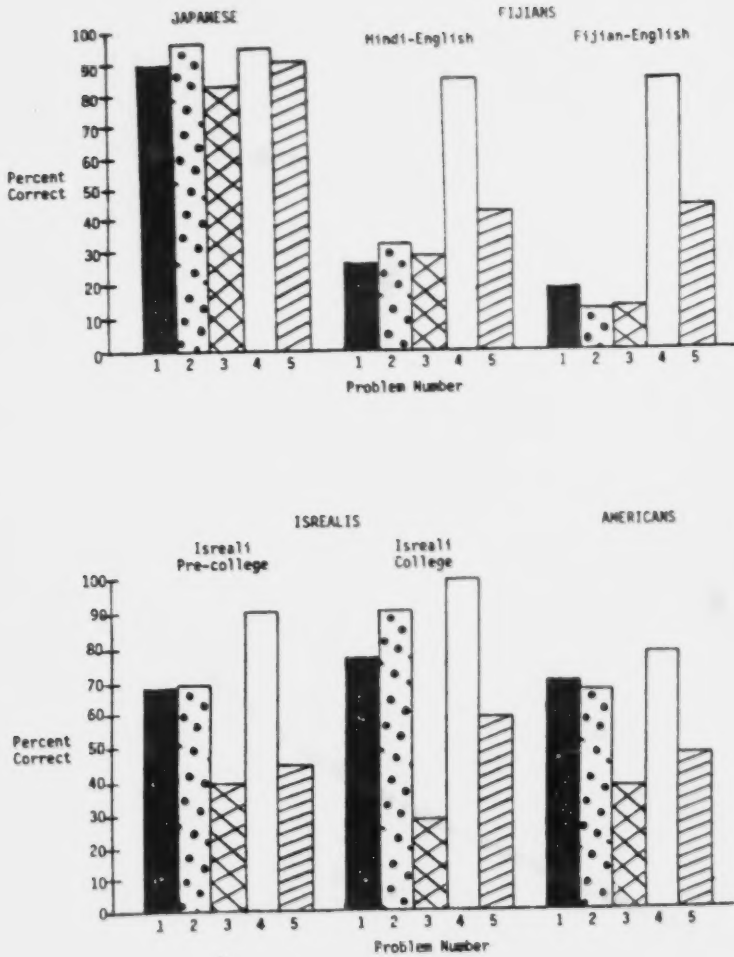
4. A store sells 5 bananas for every 7 apples. If one day they sell 35 bananas, how many apples did they sell that day?

5. The equation $A=7B$ gives the relative number of apples, A, and bananas, B, in a bowl of fruit. Does the bowl contain more apples or more bananas?

Before presenting the test results, we should point out some of the limitations with the test, and with the experimental design. First, although problem 1 is equivalent in structure to the "students and professors" problem, there is one very important distinction between these two problems that prevents detailed comparisons in performance between them. In problem 1, the variables used to represent the number of students and the number of teachers are X and Y, respectively. On the other hand, the corresponding variables in the "students and professors" problem are S and P, which also correspond to the first letter in the words "students" and "professors". There is evidence that a subject with the proclivity to use S and P as labels for "students" and "professors" may not have the same proclivity if asked to use the variables X and Y. Further, the exam was administered by colleagues who teach in universities in the respective countries, and we cannot be certain that the testing situations were reasonably equivalent for all groups. However, we do believe that the results give us an indication of the extent of some of the misconceptions described earlier for these five ethno-linguistic groups.

RESULTS

The percentage of each group obtaining the correct answer on each problem for the combined female-male population is shown on the bar graphs below. The data is presented in terms of the combined female-male population, since there were very few statistically significant sex-differences in performance. To test for sex-differences in performance, we carried out a 2 by 2 chi-square analysis between males and females, taking into account the total number correct versus total number wrong for each question. Two groups, the Israeli college group and the Fijian-English group, lacked a sufficient number of subjects in the sample to allow an appropriate χ^2 analysis. There were only



two sex-differences in performance found at the .05 significance level. These were the Japanese group in problem 1 and the Israeli pre-college group in problem 2.

In Table 1, we show a breakdown of the performance of each group by questions. Each entry represents the percentage of each group in the respective category.

Our results show that the Japanese group has the best overall performance on the test. The Israeli college group is next, and the American and Israeli pre-college groups follow, with a similar overall performance. Next are the two Fijian groups, with the Hindi-English group outperforming the Fijian-English group. Except for the two Fijian groups, problems 3 and 5 proved to be the most difficult, with the performance on problem 3 being consistently below the performance on problem 5. The two Fijian groups, on the other hand, had the most difficulty with problems 1, 2, and 3; unlike the other groups, problem 3 did not cause more difficulty than problems 1 and 2 for the Fijians. Problem 4 consistently caused the least difficulty across groups. For each of the 6 groups taken separately, the performance on problem 1 was very similar to the performance in problem 2.

DISCUSSION

Two interesting findings emerged from this study. The first is the inordinate difficulty caused by problem 3. It appears that writing an equation relating two variables is more difficult for students when they are given a pictorial description of the situation than when they are given a textual description. This result is consistent with previous findings (Mestre, et al., 1982; Clement, 1982). It may be that asking students to write an equation from a static, concrete entity, such as a picture, increases the likelihood that the student will think in concrete terms. This explanation is consistent with a popular error-type shown on Table 1, which consists of writing an equation made up of summing some linear combination of P and C, and setting this equal to a third quantity, which is often "6". The "6",

Table 1
Breakdown of Answers

Answers Prob. 1-5	Filians		Israeli		Japanese	American
	Hindu N=79	Native N=38	College N=41	Pre-Coll. N=103	N=171	N=85
X=6Y	25.3	18.4	75.6	67.0	88.3	73.5
6X=Y	62.0	52.6	22.0	32.0	11.7	16.5
6X=Y, 6Y=X, or 6X=Y=1	8.9	26.3	0.	0.	0.	7.1
Other	3.8	2.6	2.4	1.0	0.	4.7
2 gallons	31.6	13.1	90.2	68.0	96.5	67.1
4.5 gallons	49.4	55.3	9.8	25.2	2.3	5.9
6 gallons or 9 gallons	7.6	15.8	0.	0.	0.	8.2
Other	11.4	15.8	0.	6.8	1.2	18.8
C=5P	27.8	13.2	29.3	39.8	82.5	37.6
5C=P	45.6	47.4	61.0	47.6	4.1	24.7
5C+P=6, 5P+C=6, and types $\square \times \square = \square$	19.0	13.2	0.	1.0	3.5	1.2
5C:1P, 5C/1P, or C:P=5:1	3.8	10.5	0.	0.	7.0	8.2
Other	3.8	15.8	9.8	11.6	2.9	28.2
49 Apples	84.8	84.2	100.	90.3	94.2	78.8
37 Apples	3.8	5.3	0.	1.9	0.	2.4
25 Apples	1.3	0.	0.	0.	4.7	2.4
Other	10.1	10.5	0.	7.8	1.2	16.5
More Apples	43.0	46.7	58.5	45.6	90.0	47.1
More Bananas	57.0	55.3	22.0	42.7	8.8	36.5
Equal # of each	0.	0.	4.9	5.8	0.	3.5
Other	0.	0.	14.6	5.8	1.2	12.9

of course, corresponds to the total number of animals in the picture.

The second interesting finding is the performance of the American group on problem 1. Approximately 72% of these students obtained the correct answer in this modified version of the "students and professors" problem. By contrast, only 43% of a similar group of non-technical college students in the Clement, et al. study were able to obtain the correct answer in the "students and professors" problem. This is in conflict with Kaput's data (Clement, 1982) which suggests that the use of X and Y instead of S and P does not significantly alter problem difficulty. A future investigation comparing the use of variable pairs X-Y and S-P would shed light on whether the error rate is affected by using variable names (e.g. S and P) which could be confused for labels for the quantities in the problem.

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D. COMPUTERS AND MATHEMATICS EDUCATION /

L'ORDINATEUR ET L'ENSEIGNEMENT DES MATHÉMATIQUES

L'ordinateur et les mathématiques enseignées à l'école

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In this short account, we discuss the impact of the use of computers on the mathematical curriculum. To shed light on this issue, we begin by considering two related contexts. First, the past "revolutions" in the teaching of mathematics: the audio-visual era (animation films widening the viewer's mathematical experience), the "new math" reform (beware of meaningless concepts and of imbalance between rigour and intuition) and the advent of the calculators (still mostly used in very traditional ways). Then we investigate the uses of computers by mathematicians and mathematics' users: numerical calculations (mostly in applied mathematics), symbolic mathematics (especially when dialogue is established between the software and the user), source of mathematical problems (the rise of discrete mathematics) and tool for exploring mathematics (interactive computer graphics). Next, we consider changes in the mathematical curriculum, particularly in mathematical methods: model building (software can reduce the mathematical sophistication needed), exploration of mathematics (interaction and freedom), use of mathematical software (asking for what level of understanding?) and programming (mathematics with a different taste). Finally we ask some questions about "turtle geometry", computer representations of mathematical situations, building up intuition in mathematics, etc...

Dans ce bref exposé, je voudrais discuter de l'impact des ordinateurs sur l'enseignement des mathématiques. Je ne dirai presque rien des différents types d'interaction entre l'élève, l'ordinateur et le professeur, ni des effets possibles sur la nature et le déroulement des apprentissages. Je tenterai plutôt de me situer dans cette "perspective épistémologique" qui apparaît en filigrane de cette rencontre, et d'étudier les changements prévisibles des contenus et des méthodes mathématiques enseignés à l'école, à différents niveaux.

Dans un premier temps, j'essaierai de prendre un peu de recul et je me placerai, pour ce faire, dans deux contextes très différents qui jettent un peu de lumière sur le problème de l'introduction des ordinateurs dans l'enseignement des mathématiques: les différentes "révolutions" récentes dans l'enseignement des mathématiques, et les divers types d'utilisations des ordinateurs par les mathématiciens. Puis, sous ces deux éclairages, je passerai brièvement en revue ce qui s'est fait et je risquerai quelques suggestions sur ce qui pourrait avantageusement être fait pour utiliser de façon féconde l'ordinateur pour enseigner les mathématiques. Je terminerai comme il se doit par des questions et des suggestions de recherches à entreprendre.

LES REVOLUTIONS DANS L'ENSEIGNEMENT DES MATHÉMATIQUES

L'enseignement des mathématiques a connu récemment plusieurs "révolutions": l'introduction de l'audio-visuel, la réforme des mathématiques modernes, l'utilisation des calculatrices. Il nous apparaît intéressant de caractériser brièvement ces "révolutions", du point de vue disciplinaire que nous avons adopté, en faisant les rapprochements qui s'imposent avec l'avènement de l'ordinateur à l'école.

On compare souvent l'implantation des ordinateurs à l'introduction de l'audio-visuel: après tout, il s'agit dans les deux cas de l'intrusion d'une nouvelle technologie à l'école. Or on sait que la période audio-visuelle n'a laissé que très peu de traces en enseignement des mathématiques. On peut identifier deux causes principales à cet état de choses: d'une part la difficulté de préparer des documents audio-visuels valables et la pénurie qui en est résultée, et d'autre part la très faible interactivité permise à l'époque par l'équipement disponible. (Il faudrait, bien sûr, ajouter la rapidité avec laquelle on a voulu procéder et, en conséquence, le manque de préparation des professeurs. Ce phénomène semble, hélas, une donnée constante dans toutes les "révolutions".) Le premier problème reste entier quand on le traduit en termes de logiciels éducatifs. Quant au second, il peut presque disparaître si on réussit à intégrer harmonieusement ordinateur et audio-visuel.

Soulignons que la période audio-visuelle a produit certains films d'animation qui, en fournissant des représentations concrètes, pertinentes et dynamiques d'objets et de situations mathématiques, pouvaient étendre le domaine des expériences mathématiques du spectateur, et contribuer ainsi à affiner son intuition dans ces domaines. Il s'agit là d'un apport extrêmement intéressant (et rare!) de l'audio-

visuel), qui est susceptible d'être décuplé si on y ajoute une grande interactivité, rendue possible par les ordinateurs.

La réforme des "mathématiques modernes" n'a pas, dans l'ensemble, réalisé ses promesses. De nombreuses explications furent invoquées pour expliquer ce semi-échec, et nous ne retiendrons ici que deux des raisons se rapportant à notre propos: des contenus non significatifs et un niveau de formalisme trop élevé. (Pour une discussion plus détaillée, voir Dionne [1982] et [1983].)

Plusieurs concepts introduits par les "mathématiques modernes" n'ont de sens que comme synthèses de connaissances préalables, en réponse à des questions précises, de nature mathématique ou épistémologique. Par exemple, la géométrie des transformations répondait au besoin de clarifier les rapports entre les diverses géométries (programme d'Erlangen). Par contre, une telle approche de la géométrie au niveau primaire ou secondaire n'est pas significative tant pour l'élève ("la translation n'a rien transformé, puisque j'ai le même dessin") que dans la perspective d'une construction progressive de la géométrie ("pourquoi les transformations agissent-elles sur tout le plan alors qu'on ne manipule qu'une figure à la fois?", etc...). Le même problème se retrouve parfois dans certains didacticiels: verra-t-on apparaître de nouveau des mathématiques artificielles parce qu'elles sont plus facilement réalisables sur ordinateur?

Il appert que les "mathématiques modernes" ont privilégié de fait une pensée déductive, formelle, qu'on n'a pas réussi à relier à l'expérience des élèves. Par contre, l'utilisation de l'ordinateur peut faciliter une approche expérimentale, intuitive des mathématiques. Il faudra veiller à ce que l'équilibre ne soit pas de nouveau rompu, au détriment cette fois de la pensée déductive.

Passons maintenant à l'utilisation des calculatrices en classe. Les calculatrices sont d'abord apparues hors de l'école, et les professeurs ont longtemps été réticents à les accepter en classe, craignant qu'elles ne nuisent à l'acquisition des habiletés arithmétiques de base: nous savons maintenant que, lorsqu'elles sont utilisées judicieusement, elles peuvent aider à acquérir ces habiletés de calcul (voir Weaver [1981]). Leur présence massive dans les classes a même déjà contribué à des changements dans les contenus et les habiletés visés par l'enseignement des mathématiques. Par exemple, l'adoption du système métrique aidant, on insiste maintenant plus sur les nombres décimaux et moins sur les fractions dans les programmes de mathématiques. La consultation de tables et l'utilisation de règles à calcul et de techniques d'interpolation au secondaire sont devenues choses du passé. On vise maintenant par ailleurs à développer des habiletés nouvelles, reliées à l'approximation.

Toujours selon Weaver [1981], les calculatrices ont surtout été utilisées pour vérifier des calculs (principalement au primaire), faire des calculs, jouer (à des fins de motivation) et (plus rarement) faire de l'exploration numérique. Il conclut:

"It really is not surprising to find that the most common uses of calculators are relatively pedestrian ones. This should change as teachers learn about and personally explore more significant roles for calculators."

On peut se demander si les ordinateurs ne prendront pas plutôt la relève des calculatrices. Mais la question est peu pertinente compte tenu qu'il est déjà ardu sinon inutile de faire la distinction entre les super-calculatrices et les micro-ordinateurs de bas de gamme. Quant à identifier, et à fortiori mettre au point, les diverses façons riches et profitables d'utiliser l'ordinateur pour enseigner les mathématiques, disons à tout le moins qu'il reste beaucoup de travail à faire, et que cela demandera imagination et expérimentations. On peut cependant obtenir des indices en étudiant les diverses façons qu'ont les mathématiciens et les utilisateurs des mathématiques de se servir de l'ordinateur. C'est ce que nous ferons à la section suivante.

ORDINATEURS ET MATHÉMATIQUES

Les mathématiciens et les utilisateurs des mathématiques ont, depuis le début, considéré l'ordinateur comme un instrument privilégié de calcul numérique. Mais, avec le temps, ils ont réussi à mettre à profit les autres capacités de l'ordinateur (traitement de l'information non-numérique et représentations graphiques de l'information). Si bien que, de nos jours, on peut distinguer quatre types d'impacts de l'ordinateur en mathématiques: calculs numériques, mathématiques symboliques, source de problèmes mathématiques et instrument d'exploration. Dans le reste de cette section, nous décrivons brièvement ces types d'applications et nous essaierons, à la section suivante, de tirer les conclusions qui s'en dégagent concernant l'enseignement des mathématiques.

Le calcul numérique a été, et reste encore, la principale utilisation de l'ordinateur à des fins mathématiques. Nous savons, tant par des arguments théoriques (théorie de Galois des équations polynomiales, théorie de Picard-Vessiot des équations différentielles, etc...) que par nécessité pratique, que le mathématicien doit faire appel aux approximations de l'analyse numérique s'il veut trouver une solution satisfaisante à certains de ses problèmes. Le type de problème mathématique qui est amené à une solution numérique par ordinateur provient habituellement de la modélisation d'une situation concrète et vise de ce fait une application hors des mathématiques, étant donné que ce genre de solution n'est généralement pas considéré comme satisfaisante en mathématiques fondamentales. (Il y a, bien entendu, des exceptions à cette règle: par exemple lorsque l'on veut déterminer si un grand nombre donné est premier ou non.)

Depuis quelques années, des mathématiciens se sont vus contraints de faire appel à l'ordinateur pour les aider à prouver des résultats. Cette symbiose entre le mathématicien et l'ordinateur a suscité bien des débats, mais il semble bien que

l'ordinateur deviendra de plus en plus un outil essentiel pour les démonstrations. Le lecteur intéressé à en connaître plus pourra consulter Lehmer[1969], Davis & Hersh[1980], Bouvier[1981] et Spencer[1983].

Parallèlement à cela, des recherches dans le domaine de l'intelligence artificielle ont amené à la mise au point de logiciels capables de prouver sans aide extérieure des résultats mathématiques (voir Slagle[1982] et Pavelle, Rothstein & Fitch[1982]). Mais, bien qu'on ait obtenu dans certains cas des résultats intéressants, c'est encore lorsqu'on a conçu ces logiciels de telle sorte qu'un dialogue puisse s'établir avec l'utilisateur que l'on obtient des outils vraiment puissants et utiles.

Il est incontestable que l'informatique a bénéficié et bénéficie encore d'apports immenses de la part des mathématiques: qu'il suffise de rappeler le rôle de la logique mathématique dans la description même de l'ordinateur moderne, où le programme est stocké en mémoire au même titre que les données (voir Pollack[1982b] pp 18, 19 et 34). Mais il est aussi incontestable que l'informatique influence de plus en plus les mathématiciens. Voyons quelques exemples. On cherche de plus en plus à trouver ou à améliorer des algorithmes, à mesurer leur efficacité, à les comparer entre eux. On cherche à modéliser les ordinateurs (théorie des automates) et faire des preuves à propos de leurs systèmes d'opération (voir Pollack[1982a]). Et la liste des domaines fécondés par l'informatique est fort longue: théorie de l'information, théorie du codage, combinatoire, mathématiques discrètes, etc... Certains (Ralston[1981] par exemple) prévoient que les mathématiques infinies et continues perdront leur rôle prépondérant au profit d'un équilibre avec les mathématiques finies et discrètes, plus proches des ordinateurs.

Dans le dernier type d'apport de l'informatique aux mathématiques, l'ordinateur joue un peu le même rôle que l'expérimentation dans des conditions contrôlées en sciences physiques, à ceci près que, les êtres mathématiques n'existant pas physiquement, on travaille sur certaines de leurs représentations. D'abord numériques (voir Lehmer[1969] et Davis & Hersh[1980] pp. 363 à 368), les représentations se sont diversifiées, devenant parfois interactives et visuelles (voir Davis & Hersh[1980] pp 400 à 405). Bien que ces représentations données par l'ordinateur puissent contribuer à affiner l'intuition et amener à conjecturer des résultats, notons tout de même qu'elles sont pour le moment limitées aux objets et situations mathématiques dont le degré d'abstraction n'est pas très élevé.

ORDINATEURS ET MATHÉMATIQUES ENSEIGNÉES

Il est maintenant temps de revenir à la question principale de cet essai: l'utilisation de l'ordinateur va-t-elle changer les contenus et les méthodes visés par l'enseignement des mathématiques? Disons tout de suite que la très grande majorité des expériences d'introduction de l'ordinateur dans l'enseignement des

mathématiques ont utilisé une "pédagogie informatique" (cours programmés, jeux éducatifs, exercices répétitifs, programmation, etc...) pour traiter de contenus et de méthodes mathématiques traditionnels. Mais il y a de rares exceptions...

Examinons tout d'abord les changements de contenus. L'utilisation d'ordinateurs ayant des possibilités graphiques au niveau secondaire et collégial accentue la nécessité de comprendre les relations fonctionnelles et de pratiquer l'interprétation des graphiques; par contre, des activités comme les manipulations algébriques (voir à ce sujet Wilf[1982]) et le tracé de graphes fonctionnels sont maintenant beaucoup moins importantes.

On a déjà expérimenté (par exemple Frey[1982]) des approches utilisant des progiciels (dont on expliquait vaguement et de façon intuitive le fonctionnement) qui permettaient d'éviter des difficultés techniques (comme trouver des solutions d'équations de degré trois ou plus, résoudre des cas de programmation linéaire à un grand nombre de variables, etc...) pour se consacrer plus tôt à des activités de modélisation et discussion comparative de divers modèles utilisés pour résoudre des problèmes de complexité réaliste. On a aussi suggéré (voir Ralston[1981] et Hodgson[1983]) de faire plus de place aux mathématiques discrètes, aux niveaux collégial et premier cycle universitaire. Soulignons en terminant qu'on a proposé à des élèves du niveau primaire des activités d'exploration d'une géométrie "différentielle": la géométrie de "la tortue" (voir Papert [1980] et Bossuet[1982]).

Passons maintenant à l'apparition ou la mise en valeur de certaines méthodes mathématiques dans l'enseignement. Nous avons rencontré implicitement ces méthodes en discutant des contenus, et nous pouvons en distinguer quatre types: la modélisation, l'exploration, l'utilisation de progiciels et la programmation. On constatera dans ce qui suit que ces méthodes se recoupent tellement (dans le présent contexte) qu'il est difficile d'en parler séparément.

Commençons par les progiciels, qui sont des logiciels spécialisés dans l'accomplissement d'une tâche quelconque et conçus pour être aisément maîtrisés par un usager non-expert. Les meilleurs sont très interactifs et sont en mesure d'aider l'utilisateur (ne serait-ce que par des messages d'erreur compréhensibles). Citons par exemple les logiciels d'analyse numérique mentionnés à propos de l'expérience de Frey, et le logiciel MU-MATH qui permet de faire des manipulations symboliques d'expressions mathématiques sur micro-ordinateur (voir Wilf[1982]). A l'heure actuelle, les progiciels mathématiques sont nombreux, quoique de qualité inégale, et les diverses façons de les intégrer à l'enseignement des mathématiques ont été peu explorées.

La modélisation consiste essentiellement à considérer une situation posant problème, préciser la question, la traduire en un problème mathématique, solutionner celui-ci, et interpréter le résultat en termes de la situation première pour évaluer si le problème initial a été résolu. Dans les cours de mathématiques, quand

modélisation il y a, on s'attarde longuement (pour ne pas dire exclusivement) sur l'étape de résolution du problème mathématique associé. Et comme une situation le moins réaliste produit en règle générale un problème mathématique difficile, que ce soit en termes de sophistication mathématique ou de taille (volume des calculs à effectuer), on s'attaque le plus souvent à des "problèmes d'opérette". Pourtant on peut, dans certains cas, remplacer la sophistication mathématique par la puissance de calcul des ordinateurs. L'utilisation de progiciels appropriés est donc susceptible de conduire à un usage plus fréquent de la méthode de modélisation, appliquée à des problèmes significatifs.

La question suivante se pose alors: jusqu'à quel point doit-on demander à l'élève de produire, ou à tout le moins comprendre le fonctionnement des progiciels qu'il utilise? Plus généralement, on peut s'interroger sur la pertinence d'avoir recours à la programmation pour apprendre les mathématiques. Prenons un exemple: on peut voir les nombres premiers en les définissant et en les étudiant via la méthode déductive, mais on peut aussi imaginer écrire un programme pour les identifier ou les engendrer et les étudier expérimentalement. Doit-on choisir entre ces deux approches? Les deux ont-elles leur place?... Il ne faut pas perdre de vue que l'exemple d'utilisation de l'ordinateur en enseignement des mathématiques dont on parle le plus, le LOGO, est avant tout un langage de programmation avec des commandes graphiques particulières ("la tortue", qui peut et a d'ailleurs été simulée dans d'autres langages informatiques).

Il ne faut pas oublier d'ailleurs que ce qu'il est convenu d'appeler la "philosophie LOGO" (qui est en fait proche parente de la philosophie piagétienne en éducation) préconise une exploration des mathématiques par "la tortue". Cette exploration n'est pas, en principe, reliée au langage LOGO non plus qu'au niveau primaire (voir Abelson & diSessa(1980) et le compte-rendu de Francis(1983)). Le concept d'exploration des mathématiques peut d'ailleurs prendre plusieurs formes, et "la tortue" n'est qu'un des outils disponibles (voir, par exemple, Boileau(1982)). Notons en terminant que le fait d'arriver à des concrétisations pertinentes contrôlées par l'apprenant constitue l'une des perspectives les plus intéressantes ouverte par l'arrivée de l'ordinateur en enseignement des mathématiques.

CONCLUSIONS

Disons tout d'abord que l'introduction des ordinateurs dans le milieu scolaire est encore trop récente pour qu'on puisse tirer des conclusions générales. Sa venue a été massive et rapide, et déjà nous rencontrons les problèmes de manque de préparation des professeurs et de manque de logiciels éducatifs de qualité. Etant donné l'omniprésence de l'ordinateur dans la société en général et dans le monde du travail en particulier, nous savons que l'ordinateur est entré à l'école pour y rester.

Nous commençons à peine à explorer les divers types d'utilisation de

l'ordinateur dans l'enseignement des mathématiques. Il semble bien que les apports les plus prometteurs dans ce domaine consistent à l'utiliser comme une machine qu'on peut tour à tour programmer pour résoudre des problèmes et utiliser comme outil spécialisé, via des progiciels. Il semble essentiel que la plupart de ces progiciels soient fortement interactifs et demandent à être guidés par l'utilisateur: c'est là une forme de programmation, mais dans un langage plus spécialisé et plus simple. Par exemple, on peut imaginer guider la recherche par l'ordinateur des zéros d'une fonction en lui dictant la méthode à utiliser (qui pourrait être de demander à l'ordinateur de tracer le graphe de la fonction et laisser à l'utilisateur le soin de les localiser visuellement), la précision à atteindre, etc... Il faudrait laisser une place très importante aux progiciels donnant la possibilité d'explorer très librement des situations mathématiques...

L'utilisation de l'ordinateur dans l'enseignement des mathématiques n'est pas sans quelques dangers. L'ordinateur est un instrument fascinant, et il est facile de se concentrer sur le médium au point d'en oublier le message! Il faudrait toujours veiller, en particulier, à ce que le "message mathématique" reste significatif de façon intrinsèque pour l'élève, et à conserver un équilibre entre les aspects expérimental et théorique, fondamental et appliqué, de l'apprentissage des mathématiques.

Pour le didacticien des mathématiques, la venue de l'ordinateur a suscité, et continuera de susciter, beaucoup de questions. Quel rôle devrait jouer la géométrie de "la tortue" dans la formation mathématique des élèves après le primaire? Quelles sont les nouvelles représentation d'objets mathématiques mises à notre disposition par l'ordinateur, et que nous permettent-elles de faire? Peut-on utiliser l'ordinateur comme outil pour développer l'intuition? Comment se comparent les habiletés de résolution de problèmes en mathématiques et en informatique?... Un bel avenir en perspective!

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COMPUTERS AND MATHEMATICS LEARNING: CONTINUING STUDIES

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RESUME

Nous rapportons ici la suite d'une recherche sur l'impact des didacticiels dans l'apprentissage des mathématiques. Les études en cours sont centrées sur le "continuum" du mode de représentation par l'action (enactive) au mode de représentation iconique et sur les effets sur les mesures d'habiletés spatiales de l'utilisation de figures animées en enseignement de la géométrie.

Computers and Mathematics Learning:
Continuing Studies

The Technology and Basic Skills-Mathematics Project (TABS-Math) was begun in 1980 under contract with the U.S. Department of Education. The specific goals of the project were (and are) to create computer-based courseware for mathematical concept learning and problem solving at the upper elementary/middle school level. The work of the project began with the identification of particular topics for instructional development and the articulation of a philosophy to guide that effort. The keystone principle underlying the project was the idea that the computer should provide essentially new approaches to difficult mathematical ideas, and that these approaches should be grounded in a matching of computer capability with the caveats of prior research on learning. Therefore project staff felt no compunctions to follow typical textbook definitions and illustrations of concepts, but rather analyzed the mathematics of selected concepts in the light of research on concept learning (e.g. Clark 1980), problem solving (e.g. Kilpatrick 1974, Goldin and McClintock 1979, Lester and Garafoldi 1982), and analysis of misconceptions (e.g. Shaugnessey 1980) and strategies (e.g. Reys 1980, Levine 1980). In addition research on learning from games and activities (e.g. Malone 1979, 1980) was deemed highly relevant, while programmed learning was considered of only tangential relevance. The topics of instruction chosen from the Ten Basic Skill areas identified earlier (NCTM 1980) were geometry, estimation, probability and statistics, and computer literacy.

The materials developed have been submitted to formative and summative evaluation procedures in a variety of school contexts (the specific contexts and number of subjects varies with the curricular topic). The following paragraphs summarize both formal and informal methods and findings; detailed reports will be

available at the meeting. It should be noted that in each area the instructional objectives varied to some extent from the traditional school objectives, either in their global intent (see probability below) or in their operational characters. In many cases the objective of the TABS-project instruction was hypothesized to subsume the more traditional objectives; in other cases the traditional objectives (especially vocabulary) were expected to be achieved incidentally. Because the TABS project materials differ from traditional materials both in the medium of instruction and in the approach to content, identification of meaningful control groups and control treatments was not possible in the current round of studies. This can and will be remedied in the 1983-84 academic year; because computers are more widely available control on mediation is possible, and because the pressure to improve mathematics and science instruction is causing school districts to focus more directly on the topics of the TABS materials control on the approach to content is becoming possible.

Probability and Statistics. The theory of probability has two distinct roots, an empirical base and a set-theoretic base (Gnedenko 1960). While the approach to probability in most school textbooks is a theoretical one, Shaughnessey's research (1981) indicates that students develop misconceptions which could probably be attached directly with an empirical approach. The computer, with its random number generator and versatile display capability provides unique opportunities for an empirical (or at least simulated) approach to the key ideas of probability: randomness, the law of large numbers, experiment, outcome, ratio, and so on. TABS-Materials were designed to make use of these capabilities, and therefore present an essentially different approach to probability from traditional text materials. Therefore traditional evaluation techniques were inappropriate for measurement of learning from these materials, and new instruments were developed; some evaluation techniques differ markedly from traditional tests because the concepts being taught frequently

lead to different answers at different levels of sophistication. As learners experience probabilistic phenomena many move from asserting that a statistically rare event won't happen to "it happens lots" to "it happens but not too often." Data analysis to date indicates definite change on the probability measures; the meaning of change is still being interpreted.

In order to measure change in estimation skill among students using TABS materials parallels were written for each item in the estimation test developed by Reys (1980). Two parallel forms, each containing one-half new items and one-half validated items were created randomly for use as pre-E and posttests. T-tests showed highly significant ($p .001$) differences for each classroom as well as the entire population. Multivariate analyses indicate subsets of items for which gains were greatest.

Data analysis for geometry and computer literacy are currently in process. However, changes are expected, especially in the areas of symmetry, triangle inequality and area and perimeter. For the symmetry concept, in particular, pretest scores verified teacher statements that many intermediate school students did not understand the concept. After using the TABS disks for approximately two weeks, students generally reported that they were too easy and that symmetry itself was easy.

Directions for future research. In the autumn of 1983 several new rounds of testing of the TABS materials will begin; in general these studies will be more controlled than previous ones. In addition some of the materials will be used with a remedial college population. We have begun a series of studies (Damarin and Damarin 1983) to relate success on these materials to mental abilities and styles (spatial visualization, category width, pattern recognition) and have some interesting preliminary results.

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INTEGRATED SOFTWARE FOR PROBLEM SOLVING
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RESUME

De bons didacticiels en éducation, - des programmes qui exploitent les avantages de la technologie pour enseigner - doivent à la fois offrir un bon enseignement et être de bons didacticiels. Ceci exige d'intégrer le potentiel de la technologie, tout en restant sensible au développement de l'enfant et à la connaissance de la matière enseignée. Les développements récents dans les logiciels appliqués dans le domaine des affaires fournissent des schémas pour l'élaboration de didacticiels éducatifs. Les logiciels intégrés utilisés dans les affaires constituent un environnement où le travail et la résolution de problèmes peuvent être réunis dans un même micro-ordinateur.

Dans le présent exposé nous décrivons le développement d'un environnement intégré d'apprentissage qui fournit une variété d'outils expressément conçus pour l'utilisation en classe. Cet environnement est un médium pour l'exploration et la créativité puisqu'il place l'ordinateur sous le contrôle de l'élève et qu'il permet des activités normalement inaccessibles, comme celles impliquées dans la résolution de problèmes du "monde réel".

Le système évolue autour d'un programme de données de base (Notebook) et inclut des outils pour l'analyse numérique, ainsi que des logiciels pour le traitement de texte et le dessin de graphiques et de cartes. Dans la classe, ce système dépasse les frontières disciplinaires préconçues et procure des moyens faciles pour écrire des informations numériques, ou pour faire des calculs à partir de matériel écrit.

Dans la présente communication nous discutons de deux qualités d'un bon didacticiel. Plus précisément, où en est-on rendu dans la technologie des didacticiels, et quel est l'état des choses en éducation?

Good educational software -- programs that take advantage of the technology to teach -- must be both good education and good software. This requires the integration of the potential of the technology with a sensitivity to the development of the child and a knowledge of the subject area. In this paper we discuss these two qualities separately. Namely, what is the state of the art in software technology, and what is the state of the world for education?

INTEGRATED SOFTWARE -- STATE OF THE ART

Software development is segmented into business, game and educational applications. It is unfortunately the case that most state-of-the-art programming is being done in the business and game areas. A relatively small group of knowledgeable and talented programmers have focused on developing business and game software -- as opposed to educational software -- and it is to these programmers that educators should turn for direction. Developments in these areas provide a basis for projecting future developments in educational software. The game programs have clearly anticipated the graphics and animation of the future. In this paper, we argue that recent developments in business software also provide patterns for the development of educational software.

In 1981 Software Arts, the people who created VisiCalc, established a standard format for the exchange of information between personal computer programs. DIF, standing for "Data Interchange Format", was created to solve one of the most frustrating problems facing users of application software: the need for time-consuming re-entry and modification of data. That is, as explained by Kalish and Mayer, "Users often know that one program has stored data with which a second program could work wonders, but incompatible storage formats make the data inaccessible to the second program."

Using DIF you can assemble data in a database system (DB Master(TM) or Visifile(TM)), send it to a spreadsheet

(VisiCalc(TM)) or to TK'Solver(TM) and manipulate it, then create a graphic display (using Visi-trend/Visiplot(TM) or Trendspotter(TM)). Thus, several different sorts of programs are integrated. They share data, that is, although these programs are too large to be co-resident in a microcomputer, the results of computations can be passed freely from one program to another. They provide a unified business environment for working, and problem solving, all in one microcomputer.

The power of this idea is made even more evident when one looks at the excitement and rhetoric generated by Apple's release of LISA, standing for "Local Integrated Software Architecture". In Apple's own understated announcement, Lisa is touted as "... a new kind of personal computer aimed at revolutionizing the way work is done in office environments." Lisa, with a megabyte of memory and over \$50 million and more than 200 person-years in development, goes somewhat beyond the limits of the micro and is a bit beyond the resources of the typical educational software developer, however the design concepts are worth examining.

Lisa was developed to "uniquely address the needs of individuals who work in offices." There are six integrated software applications that cover the core functions of today's offices: spreadsheet analysis, word processing, business graphics, graphics design, personal filing, and project management. These are all available to the user and information may be transferred between them quickly and easily.

But we are not writing this paper to advertise the advantages of Lisa. Rather, in my judgment, DIF, first, and Lisa, even more so, underscore the need for educators to ask a series of parallel questions regarding the school microcomputer environment. Lisa, with six integrated programs, is designed to address the needs of individuals who work in offices. How in fact might we design similar systems for children in the schools? What is the integrated software that covers the core of

today's schools requirements? How might we integrate several programs so that we can provide an environment for real-world problem solving in today's classroom?

INTEGRATED LEARNING

As stated earlier, good educational software must be good education, as well as good software. Good education creates situations in which learning can occur, and good teaching is setting up situations in which students can discover and explore. These situations are exemplified in "real world" problem solving, as described for example in Lesh 1982. Lesh characterizes these situations as follows: "The definition of problems, givens, and goals all required input from the student.... The goal was not simply to arrive at a mathematical answer; mathematics was a means to an end, not an end in itself." (p.182)

"Real world" problem solving is complex. An integrated learning environment provides a variety of tools that simplify the process by performing some of the procedural, repetitive, tasks, and allowing the student to focus on conceptual or modeling issues. Lesh describes one problem where he asks students, split into groups of three, to pretend that they are counselors at a sports camp. "Their job was to sort twelve incoming campers into three groups using any or all of the following information which was available for each camper: 50 yd. dash times, 100 yd. dash times, high jump distances, long jump distances, swimming levels, and coaches recommendations."

There is no clear answer to this sort of problem, and it is characterized by a dependence on mathematical calculation, and the manipulation of data. The heart of the problem is brought out in the variety of the models that the students used to order, or rank, the campers.

Lesh found that the students alternated between several interpretations or models of the problem -- there were four basic models that the students used -- from a

procedural perspective they involved adding, comparing, or assigning weights to different scores. In some cases this would result in a trial solution based on part of the information, and would require refinements and additional computation. H. and large, Iesh reports that the successful groups "spent relatively more time thinking about ways to think about the problem (both goals and givens) rather than ways to do it. Conceptual considerations seemed to be far more productive than procedural considerations..." (p. 185-6).

How could a computer have aided in the solution? First, there is the data. The problem itself has several different lists of scores. These need to be sorted, resorted, and manipulated in various ways. The data for the problem could be easily handled by a small database system. The system, in an integrated environment could pass selected data to a "work sheet" for initial formatting and evaluation. It could then be passed to a numerical analysis tool -- perhaps a simple spreadsheet -- and after the calculations, it could be passed to a chart sheet, and then, for presentation, to a graphing sheet. Finally, and most important, if the results seem inappropriate, all, or some, of this could be done again.

The main benefit of the electronic learning environment is that the time and energy devoted to conceptual considerations would indeed be increased. The real problem for the student is setting up the problem -- formatting the file in the database, structuring the worksheet, setting the form for the spreadsheet, formatting charts and graphs.

Specifications for integrated software designed exclusively for the classroom are very different than an office environment. Any five of the programs that use DIF require an investment of approximately \$1000 for the software alone, and in excess of 1500 pages of manuals to read. There is confusion for the user because the different programs have different user interfaces. And, the communication between the programs is slow and

somewhat clumsy. This is in part a function of the complexity of the programs involved, in part a function of the copy protection, and in part a function of the fact that different programs were developed by different companies. Programs designed for classroom use need to be easy to use, and easy to learn to use, with only minimal dependence on manuals or extensive written materials. They should be fast, and need not be anywhere near as large, nor as complex, as similar office programs. User interface needs to be uniform and consistent, and help pages pertaining to the specific part of the program in use should be readily available.

NOTEBOOK ENVIRONMENTS

At Window, we are in the process of developing such an integrated environment. We began with Notebook, a simple yet flexible database program. What makes it appropriate for children? Notebook is inexpensive. It is not copy-protected, and automatically saves and back-up files, therefore offering additional levels of protection for preserving student files. Options are always clear (return or esc). It is fast, using memory-dependent, as opposed to disk-dependent, working storage. It is a simplified menu-driven system. Prompts -- to insert disk, turn on printer, make back-ups, for example -- are simple and frequent. Yet Notebook is flexible enough for students to create, sort, search, print-out and update their own files. Other parts of the system include a system of worksheets for interfacing between programs. There are numerical analysis tools -- a small spreadsheet, a simple calculator -- charting and graphing tools.

Pieces of the integrated system can be replaced or adapted for different age and ability levels. Thus, we are developing a database system for the primary levels. This would use larger type fonts, handle small amounts of data, and even manipulate symbols or pictures. The numerical analysis tools for this age group provide support for counting and representing patterns and

numbers. The system becomes even more powerful when you consider the implications for building a truly integrated curriculum.

NOTEBOOK AS A WRITING TOOL

Before it's release, Notebook was pilot tested with a group of sixth-grade students in a community just outside of Boston. The teacher and students, together, developed a variety of files for classroom use. These included a catalog of individualized spelling lists, a file for storing book reviews, and a tablet for sending messages. But the most popular way of using Notebook was as a tool for conducting informal, albeit original, research. Research took the form of an "Interest File," in which each student entered and updated information about themselves; e.g., their favorite rock group, their favorite subject, and the top score on their favorite videogame.

Original research of this sort is more than just an extremely motivating way for students to learn how to use a database; it is a vital means of providing real data on which to base expository writing. This is supported by the results of the National Assessment which documented that today's high school seniors are ill-prepared to meet the demands of college English (NIE 1975). The assessment prompted educators to look at the way in which writing is taught in the public school. The focus in the early years is on creative writing and story-telling. Typically, expository writing is not addressed until junior high school, at which time students are asked to fill in the blanks of a rigid five paragraph essay. Rarely are they let in on the process of organizing (and reorganizing) information and ideas to form a logical argument, leading to a defensible conclusion. Yet this is the stuff of college, and in fact, real world writing.

In developing a writing curriculum for use with Notebook, the original goal was to plan research projects in which students could collect real data, manipulate it, and then

send it directly into a word processor. The challenge would be for each student to form an opinion on an emotionally charged subject which could in some way be supported by the facts at hand. In this environment, two students could take entirely different viewpoints, yet selectively choose data to support their conclusions.

One unit we developed is a study of the videogame playing behavior of boys and girls in fourth through sixth grade. Students interview, code and enter data into the class videogame file. They are then each asked to arrive at a conclusion on the basis of the survey. Possible viewpoints that might be defended include:

- o People who spend a lot of time at the arcades have less time for studying so they become poor students.
- o Frogger is a fast-paced game that captures people's imaginations.
- o For the games girls like, they are better than boys.

Surprisingly, it was not until we began developing writing samples to demonstrate the potential of the research tool that the importance of integrating numerical and word data became apparent. Thus our integrated learning environment serves to cut across pre-conceived disciplinary boundaries, and provides an easy means to write about numerically-based information, or to make computations about written material.

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A CONSIDERATION OF AGE-RELATED TASKS IN MICROCOMPUTER LEARNING

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RESUME

Soixante-deux élèves âgés de neuf et quatorze ans ont été inscrits dans un cours d'informatique pour débutant, ou pour élèves avancés à l'intérieur du programme d'éducation permanente d'un collège local. Les études de cas de ces élèves semblent indiquer qu'il existe plusieurs stades, reliés à l'âge, dans l'apprentissage de la programmation portant sur des textes en langage BASIC. Ces stades semblent être les suivants: 1) travailler avec des énoncés faisant intervenir la commande PRINT; 2) utiliser des variables; 3) écrire des énoncés où interviennent des variables; 4) utiliser des énoncés contenant la commande INPUT; 5) écrire des boucles avec FOR-NEXT; 6) écrire des énoncés à l'aide des commandes FOR-THEN; 7) travailler avec des compteurs; 8) écrire des énoncés avec les commandes READ-DATA; 9) écrire des boucles emboîtées; 10) travailler avec des variables indicées; et 11) travailler avec des matrices. Les enfants non encore en cinquième année n'ayant pas encore atteint l'âge de onze ans, sont capables de programmer une variété d'énoncés avec PRINT pour des variables numériques ou littérales. Mais ils semblent éprouver de la difficulté avec la plupart des autres structures. L'utilisation du jeu de rôle précédemment à l'écriture d'un programme améliore l'habileté de certains enfants de cinquième année à manier les compteurs, les boucles FOR-NEXT et le dialogue interactif. Le fait de travailler avec des problèmes logiques, dans le cadre d'un laboratoire de mathématiques, augmente les chances de succès des enfants de onze et douze ans en ce qui a trait à l'utilisation des boucles emboîtées et des drapeaux.

Although much of the literature suggests using computers with elementary school pupils few studies have considered how pupils learn programming at various ages.

PURPOSE

The purpose of this study was to examine some of the computer tasks which could be accomplished by pupils between the ages of nine and fourteen who were receiving formal instruction in computer programming in BASIC. By examining the work of a variety of pupils the following questions were considered: 1) Are there specific computer constructs which can be mastered by all pupils in the upper elementary and middle school grades? 2) Are there additional constructs which can be understood with support experiences in addition to the actual work at the computer ?

SUBJECTS

The subjects were sixty-two pupils enrolled in the Young People's Computer Program, a continuing education program at a public college. Pupils were from many different schools and communities in the Bronx and lower Westchester in New York and ranged in age from nine to fourteen years old. They were grouped into beginning or advanced classes depending upon their previous experiences. Tables showing the distribution of the pupils by computer class and age follows:

Distribution of Age-Group by Class - Advanced						
PUPIL'S AGE						
CLASS	9	10	11	12	13	14
Advanced 1	1	1	3	1	4	4
Advanced 2		2	3	4	2	1

Distribution of Age-Group by Class - Beginning

CLASS	PUPIL'S AGE					
	9	10	11	12	13	14
Beginning 1		5	4	2	2	1
Beginning 2		1	5	2		
Beginning 3	1	2	2	5	3	1

METHOD

Computer classes were organized for pupils from ten to fourteen years old and advertised throughout the Bronx and Westchester. Classes met for eight sessions one day a week either after school or on Saturday or Sunday morning for one hour and fifteen minutes. All classes were held during March, April and May, 1983.

Instruction for both the beginning and advanced courses was on the Apple II plus microcomputer in a special facility in the microcomputer center of the college. Each pupil worked alone at a computer, although in one of the advanced classes, some of the pupils asked to work together on projects at some of the sessions. Four teachers were involved in the program, two junior high school teachers, one high school teacher and one college instructor. Each teacher had a minimum of ten years teaching experience in mathematics and one year of teaching BASIC on a microcomputer. Two of the beginning classes also had a student teacher assigned to assist with instruction.

It was agreed that a lesson would consist of a new technique or concept, challenge problems which were based on the new technique and projects which integrated previous learnings. The class went through the new technique as a group; individuals worked on challenges and projects with the teacher assisting as a facilitator

for the remainder of the class. Because of the span of ability and age, frequently each pupil was working on something else.

Eighteen of the pupils had microcomputers at home or in a school laboratory and frequently came to class with additional work which they had done outside of class.

Five pupils from the advanced course enrolled in an additional problem-solving course during the first two weeks of July, 1983. The purpose of the course was to explore problems which could provide some of the experience necessary for problem solving on the computer.

RESULTS

The following is an analysis by concept and age of the mastery level with and without support. Support means work with simulations, role-playing, tutoring by the instructor using a memory board, and games specifically aimed at teaching keyboard techniques.

Analysis of Computer Constructs and Age - With and Without Support Techniques

CONCEPT	AGE					
	9	10	11	12	13	14
PRINT statements	2	11	17	14	11	7
RUN vs LIST	0	9	17	14	11	7
*	2	2				
PROGRAM vs	0	7	16	12	11	7
EXECUTION*	2	4	1	2		
USE OF VARIABLE	0	7	15	13	11	7
*	2	2	2	1		
VARIABLE STATE-	0	4	15	14	11	7
MENT *	2	7	2			

Analysis of Computer Constructs and Age - With and Without Support Techniques (continued)

CONCEPT	AGE					
	9	10	11	12	13	14
INPUT statements	0	3	9	11	11	7
*	0	6	8	3	0	0
FOR-NEXT loop	0	3	8	10	10	6
*	0	7	9	2	1	1
IF-THEN statements	0	4	8	9	10	6
*	0	6	8	3	1	1
Counters	0	8	8	9	10	7
*	0	1	8	4	1	0
READ-DATA	0	7	8	9	10	6
*	0	2	8	4	1	1
NESTED LOOPS	0	1	5	9	10	6
*	0	6	9	5	1	1
SUBSCRIBED VARIABLES	0	1	2	4	6	6
*	0	5	7	5	4	1
MULTIPLE ARRAYS	0	1	2	3	6	6
*	0	2	4	5	3	1

* With support

RESULTS OF EXTRA INSTRUCTION IN PROBLEM SOLVING

Aisha

Aisha, an eleven year old student who had mastered variable and INPUT statements but who could not work with FOR-NEXT loops or IF-THEN statements was given practice in logic problems with two or three variables. At the conclusion of the eight sessions, she programmed several of the logic problems with teacher assistance. She still had difficulty with the use of an IF-THEN statement as a flag and with counters contained in FOR -NEXT loops.

Oral

Oral, a fourteen year old student who had mastered the FOR-NEXT loop and IF-THEN statements, was able to handle nested loops easily and began to work with subscripted variables. He became proficient at logic puzzles and using the computer to generate dimensioned lists. However, he manually scanned the lists for data rather than use multi-dimensional arrays.

Sergio

Sergio, an eleven year old student, learned to handle FOR-NEXT loops to generate lists and multiples. He manually scanned the lists. With a combination of an IF-THEN statement, he was able to use two variables to shorten his lists.

Franz

Franz, a ten year old student, learned to handle nested loops and IF-THEN statements to generate lists with several variables. He became very proficient at solving logic puzzles and made up several of his own to run on the computer.

Lauren

Lauren, a fourteen year old, learned to handle one-dimensional arrays and became proficient in the solution of logic puzzles. Lauren did not complete the eight session course.

CONCLUSIONS

It seems that children under eleven years of age have difficulty mastering many of the constructs of computer programming in BASIC. They seem to be successful with handling PRINT statements and manipulating PRINT statements with such commands as FLASH and INVERSE and punctuation such as "," and ";" however, it is recommended that additional instruction be delayed until they are more mature.

Eleven and twelve year old pupils benefit through the use of simulations and role-playing of computer programs. Techniques such as the Memory Board (Feddern, 1980), the Robot Walk (CEMREL, 1976), and other games can be used to enhance instruction. Direct instruction in problem solving techniques such as in logic problems and problems which can be solved through diaphantine equations, seem to assist learning of looping and logic techniques.

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E. EARLY ARITHMETIC (K-3) /

LES DEBUTS DE L'ARITHMETIQUE (M-3)

THE DEVELOPMENT OF CHILDREN'S INFORMAL ADDITION

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RÉSUMÉ

L'étude examine, durant une période de neuf mois, l'habileté à additionner de 17 enfants d'école maternelle afin d'étudier 1) les premières stratégies d'estimation, 2) la transition de l'addition faite de manière concrète à l'addition mentale, 3) la séquence de développement des stratégies d'addition mentale, 4) le rôle du principe de commutativité dans le développement de l'addition. Les résultats indiquent que, même avant d'avoir développé des procédures mentales de dénombrement pour calculer des sommes, les jeunes enfants sont capables d'utiliser la représentation mentale de la droite numérique pour faire des évaluations raisonnables de problèmes tels que $2 + 4$ et $3 + 7$. A peu près la moitié des sujets sont arrivés à faire la transition entre l'addition faite à l'aide d'objets et l'addition mentale. Les sujets ont généralement trouvé plus facilement la solution à des problèmes du genre $1 + N$ qu'à des problèmes plus complexes. L'usage des doigts, surtout la représentation automatique des nombres au moyen des doigts semblait faciliter la transition pour certains des enfants. La séquence la plus courante des stratégies d'addition mentale était la suivante (par exemple, pour l'addition $2 + 3$): d'abord compter à partir de 1 en commençant par le premier terme (1, 2; 3, 4, 5), puis compter à partir de 1 mais en commençant par le terme le plus grand (1, 2, 3; 4, 5) et enfin compter à partir du terme le plus grand (3; 4, 5). La procédure qui consiste à compter à partir du premier terme n'a été que très rarement observée probablement parce que cette procédure ne minimise pas la difficulté du double comptage exigé par les stratégies d'addition mentale. Enfin, bien que la plupart des sujets qui ont inventé des stratégies sans tenir compte de l'ordre des termes appliquaient le principe de commutativité, la connaissance de celui-ci ne semblait pas une condition nécessaire à l'invention de stratégies plus économiques.

Children typically start school with some informal addition ability. Specifically, preschoolers can usually use their mental number line to determine the effects of "adding one more"—at least for numbers up to four (e.g., Gelman, 1971). However, young children are less successful in mentally solving problems such as $1 + 3$, $3 + 2$, or $2 + 4$ that do not involve next number relationships (e.g., Gelman, 1977; Starkey & Gelman, 1982). To solve such "big" problems, the child must learn counting-based algorithms that, at least initially, require the aid of countable objects (Resnick, 1983).

What happens though when children are asked to solve big problems before they have learned a mental algorithm or the basic facts? Do they resort to guessing or engage in problem solving to produce reasonable estimates? Since preschoolers usually have an implicit understanding that addition increases numerosity (e.g., Starkey & Gelman, 1982) and at least the rudiments of a mental number line (e.g., Resnick, 1983), even kindergarteners should be capable of estimation or "protoaddition"—primitive mental addition. A relatively unsophisticated approach would involve selecting the number after the augend in the count sequence or counting up to the augend and then counting once more (e.g., $2 + 6$: "3," or "1, 2; 3—3"). A somewhat more sophisticated strategy would entail adding one to or counting just beyond the larger term (e.g., $2 + 6$: "7," or "1, 2, 3, 4, 5, 6; 7—7"). Note that such a strategy would produce correct answers in the case of $1 + N$ ($N + 1$) problems. The most sophisticated approach would differentiate between $N + 1/1 + N$ and big problems. That is, the child would realize that big problems could not produce the same sum as adding by one problems and so would use an "add several" strategy for the big problems (e.g., " $4 + 3$ can't be 5, since $4 + 1$ is 5; so maybe it's 6 or 7"). Note that this estimation strategy will often produce answers for big problems that are only off by one or even correct. One focus of the study, then, was to determine whether or not young children commonly employ estimation or primitive addition strategies and which strategies they tend to employ.

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To produce exact sums for big problems, children first rely on concrete counting-all: counting out a number of objects or fingers for the augend and the addend and simply counting the total. Later then invent increasingly elegant mental addition algorithms (e.g., Ilg & Ames, 1951). The most basic mental procedure, counting-all starting with the first addend (CAF), entails starting with "one," counting up to the cardinal value of the augend, and then continuing the count a number of steps equal to the cardinal value of the addend (e.g., $2 + 4$: "1, 2; 3[is 1], 4[is 2], 5[is 3], 6[is 4]-6"). That is, the addend is enumerated as the child counts on from the augend. Thus CAF is a fairly sophisticated invention because it does not directly model the concrete counting-all process and because it entails a double count (Baroody & Gannon, 1983b). Counting-on from the first addend (COF) short-cuts the CAF procedure by starting with the cardinal value of the first addend (e.g., $2 + 4$: "2; 3[1], 4[2], 5[3], 6[4]-6"), but it does not reduce the number of steps in the cognitively demanding double count. The double count can be minimized by starting with the larger term. One strategy that accomplishes this end is counting-all starting with larger term (CAL), which entails starting with "one," counting up to the cardinal value of the larger term, and then counting-on from there while the smaller term is enumerated (e.g., $2 + 4$: "1, 2, 3, 4; 5[1], 6[2]-6") (Baroody, in press-a). Counting-on from the larger term (COL) short-cuts this process by starting with the cardinal value of the larger terms and hence is the most economical informal mental addition algorithm (e.g., $2 + 4$: "4; 5[1], 6[2]-6").

The second focus of the study addressed issues regarding the development of children's informal addition algorithms. How readily do children make the transition from concrete counting-all to mental procedures? What facilitates this transition? Is there a typical course of development for mental algorithms? In other words, is CAL or COF the more common transitional step to COL? Finally, do the invention of strategies that disregard addend order so as to minimize the double count imply an appreciation of commutativity (e.g., Resnick & Ford, 1981)? For example, does a child who uses CAL to compute $2 + 4$ really appreciate that $2 + 4$ and $4 + 2$ are equivalent (produce the same sums)?

METHOD

The study examined the addition ability of 17 working- to upper middle-class kindergarten children (10 girls and 7 boys, initially 5 - 1 to 6 - 7 years-months, mean age 5 - 5) over 14 sessions spanning 9 months. After a familiarization phase, the subject pool was individually tested on enumeration, numeral recognition, production of a specified N and number comparison (e.g., 3 vs. 2, 5 vs. 6, 7 vs. 6, 9 vs. 10) skills as well as screened for informal addition ability. All tasks used a game format to maximize performance. The addition evaluation consisted of presenting the subject with a 5 x 8 card on which a horizontal addition problem was typed. The experimenter also read the problem to the child (e.g., "This says five and one. How much are five and one altogether?"). The experimenter instructed the subjects to solve the problems any way they wished to: using fingers, blocks, or their head. Six trials were presented in the following order: $5 + 1$, $3 + 1$, $4 + 2$, $3 + 2$, $5 + 3$, and $4 + 3$. The child's solution strategy and accuracy for each trial was noted. Subjects who did not have an organized informal strategy were taught concrete counting-all. From a subject pool of 27, subjects with the following addition ability were chosen: two girls who already exhibited mental procedures even for big problems ("advanced subjects"), five children with high learning potential (could mentally solve $N + 1$ problems and had to be shown concrete counting-all only once for big problems, or already used concrete counting-all for all problems), five with medium potential (had to be shown concrete counting-all several times and typically did not use mental procedures for $N + 1$ problems), and five with low potential (had to be shown concrete counting-all on all or nearly all trials).

In succeeding sessions, the experimental addition task was administered in a manner similar to the screening task. It consisted of 10 randomly presented trials ($1 + 3$, $1 + 5$, $1 + 7$, $2 + 3$, $2 + 4$, $2 + 6$, $3 + 4$, $3 + 7$, $4 + 5$, and $4 + 8$). Smaller term first problems were used in order to differentiate between CAF and CAL or COP and COL. Subjects were encouraged to use mental

procedures—even guessing—instead of or before they used a concrete approach. Low potential subjects were not administered the experimental task until they achieved criterion (success on at least 5 of 6 trials) on the screening task. It took one child 2, three 3, and a fourth 4 additional sessions to achieve criterion. Subjects were also checked on number comparison until they achieved criterion (at least 7 of 8 pairs correct, $p < .035$, Sign test). In order to avoid the possibility of influencing addition development, a commutativity task was administered once CAL or COL had become the predominant strategy. The commutativity task consisted of having the child compute the sum for the first problem of a pair. The second problem was then written directly beneath the first and the child was asked if that problem would produce the same answer as the one just computed or something different. Two commuted ($4 + 5$ to $5 + 4$ and $6 + 4$ & $4 + 6$) and two noncommuted trials ($3 + 7$ & $3 + 4$ and $5 + 3$ & $2 + 3$) were administered in random order. In the last experimental sessions, all subjects were administered the commutativity task.

RESULTS AND DISCUSSION

Early Estimation Ability

Though it varied somewhat by level of learning potential, the kindergarten subjects generally generated reasonable estimates—i.e., estimates which were consistent with the basic principle that addition increases numerosity. Over the course of the study, only two subjects ever gave estimates for experimental trials that were equal to or less than the augend (e.g., "3" for $3 + 7$ or $4 + 5$), and this occurred only rarely (on less than 3% of their trials). Indeed, all subjects typically gave estimates that were greater than the larger term. Four did so 100% of the time on both $1 + N$ and big experimental problems; nine at a 90% level. The six least consistent subjects (2 high, 1 medium and 3 low potential children) gave reasonable estimates at least 75% of the time. The inconsistencies of this last group of subjects may well be due, in part, to viewing addition as a unary operation—as a change of state problem. That is, because they view a problem such as

3 + 7 as "3 and (7) more," estimates of 4 or 6 may have seemed natural to them.

The children did not appear to have a single scheme for making estimates for big problems, but generally switched between adding one and adding several to the larger term. However, most children did use one strategy twice as often as the other. Moreover, strategy selection varied by group: all high learning potential subjects used an adding several strategy a majority of the time, all low potential subjects tended to add one to the larger addend, and medium potential subjects varied in their preferences.

Over the course of the study, only two subjects (one high and one medium potential child) used the adding several strategy selectively—i.e., used the strategy with big problems when it was appropriate but refrained from using it with $1 + N$ problems where it was inappropriate. By the end of the study one medium potential and one low potential child began using an add several strategy selectively. Selective use of an adding several estimation strategy may indicate that an informal unary view of addition has been supplemented by a relatively sophisticated binary view. That is, if a child views addition as the union of two cardinalities, estimates of 3 (adding one to the augend) or even 5 (adding one to the larger term) for $2 + 4$ should be rejected because they only take into account the cardinality of one term. Moreover, an adding several solution for $1 + N$ problems would be avoided because it disregards the cardinal number "one."

The Development of Mental Algorithms

Typically, the transition from concrete to mental addition was slow—even for $1 + N$ problems. Of the 13 subjects who initially relied on concrete counting—all to solve the $1 + N$ problems, only 7 (54%) shifted to using mental strategies by the end of the study. Moreover, except for one case, this transition did not occur until the eighth session or later. Somewhat less than half (7 of 17 or 41%) made the transition for big problems, and this took at least 7 sessions for four of these subjects.

Finger models appeared to play a role in the transition from concrete counting-all to mental algorithms for a number of children. In several instances, children relied on automatically representing each term on their fingers and counting the total just before they began relying on mental algorithms. In several other cases, children would enumerate one term on their fingers and then use their fingers for a different function: to keep track of a double count. For $2 + 4$, for example, transitional counting-all starting with first, designated by [CAF], involved counting as two fingers were extended, and then counting-on "3, 4, 5, 6" as four additional fingers were extended to keep track of the addend. A more sophisticated version is transitional counting-all starting with the larger, designated by [CAL] (e.g., $2 + 4$: "1, 2, 3, 4 [as four fingers are extended], 5, 6 [as two fingers are extended]—the answer is 6").

The children achieved advanced mental strategies in various ways. For big problems, one subject's intermediate strategy between concrete counting-all and COL was unknown. (The subject became very secretive during the transitional phase.) For two subjects, CAL was the intermediate strategy. For a fourth subject, both CAF and CAL preceded COL. Finally, COF appeared briefly as the transitional strategy for a fifth subject. This transience may well be due to the fact that COF does not minimize the cognitively demanding double count (Baroody & Gannon, 1983b). (In fact, COF was observed rarely—most often with "near-doubles" such as $2 + 3$, $3 + 4$ and $4 + 5$, where minimizing the double count mattered the least and picking the larger term was relatively demanding).

Consistent with earlier findings (Baroody, 1982; Baroody & Gannon, 1983a), an appreciation of commutativity was often implied by—though not a necessary condition for—the invention of addition strategies that disregard addend order. Four children who invented [CAL], CAL and/or COL, appreciated commutativity, three did not. The latter apparently adopted labor saving strategies without appreciating the implications of their actions.

Like much other recent work, this study indicates that children just beginning school have many informal addition strengths. However, these results also suggest that care must be taken not to overestimate how systematic this knowledge is.

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STRATEGIES USED BY FIRST TO FOURTH GRADE CHILDREN
TO TRANSFORM STATEMENTS CONTAINING AN EQUAL SIGN

Maurice Bélanger and Stanley Erlwanger

RESUME

Cette étude fait suite au document de Erlwanger et Bélanger publié dans le présent volume. Nous avons examiné les stratégies de transformations en calcul de 527 enfants de la première à la quatrième année de niveau primaire. Les résultats démontrent qu'il existe différents types de transformations selon l'interprétation que l'enfant donne au signe d'égalité.

In the paper by Erlwanger and Bélanger, also in this volume, we have described a study of childrens' interpretations of the equal sign. Addition and subtraction statements, written in various horizontal forms, were presented to children in first to fourth grade. They were asked to judge each statement as "correct" or "wrong". Further, if they believed a statement to be wrong they were also asked to "fix" the statement by rewriting it in a separate space provided next to each item. This enabled us to examine the "corrections" made by the 527 pupils in the study. Out of the 527 pupils, 132 were also interviewed using the same statements and this gave us further confirmation on their identification of statements they considered correct or incorrect, the basis on which they judged each statement, and finally, their strategies for "fixing" incorrect statements. This paper is a short qualitative resume of some of the strategies employed by children on selected groups of items used in the study. We intend to publish a detailed analysis concerning all the items at a later date.

In the Erlwanger and Bélanger summary paper, we stated that the children in our study appeared to fall into two categories: first, a group we called "computers", who interpret the equal sign as a separator/operator, and a second group we called "pattern-seekers", who interpret the equal sign both as a separator/operator and as an equivalence relation in terms of its reflexive property. In order to judge an arithmetic statement as correct or wrong, computers view the statement as calling for an action to calculate. On the other hand pattern-seekers simply examine the statement in terms of a pattern-form without any need to calculate. The analysis of the strategies children use to "fix" statements judged to be wrong needs to take into account whether

the child's general mathematics behavior on all the items as a whole indicates that a particular child performs as a computer or as a pattern-seeker. Each subgroup will not only select different items as correct or wrong but each respective subgroup will also employ different strategies to "fix" the statements considered to be wrong.

Only a partial qualitative analysis can be given here since an analysis based on all the items used would greatly exceed the limitations of space.

A. Common strategies used by "computers" and "pattern-seekers".

Computers and pattern-seekers consistently judged statements written in the form $a+b=c$ or $a+b+c=d$ to be correct. If an incorrect statement such as $6+4=2$ is presented, the vast majority would judge this to be wrong and "fix" the statement by rewriting $6+4=10$. Since Canadian textbooks contain a plethora of statements in this form, children already have had considerable experience with these and they appear to cause little difficulty. A few children would "fix" the statement $6+4=2$ by changing the operation, i.e. $6-4=2$, however, the vast majority changed "the answer". If the statement was written in the form $c=a+b$ or $d=a+b+c$ (e.g. $5=3+2$) both the computers and pattern-seekers generally accepted these as correct. In the interviews, they would make comments such as, "well, it's backwards, but that's alright... I'll just do it the otherway around", indicating with their finger that one could proceed from right to left to "get the right answer". Whether such statements were written in the usual school form ($a+b=c$) or "backwards" ($c=a+b$) seemed to cause little difficulty.

As mentioned in the Erlwanger-Bélanger paper, both computers and pattern-seekers interpret the equal sign as a separator/operator; if the statement is written $a+b=c$, they view this as indicating the "problem" on the left, the "answer" on the right. If written backwards, then the "problem" is on the right, the "answer" on the left, and hence all that is required is "to do it the other way around". In both cases, the equal sign is interpreted as a separator/operator, so that the only "fixing" strategy that is required in an incorrect statement is to either change the answer or to change the operation.

However, on textbook statements such as $a+b=a+b$ or $a+b=b+a$, the simplicity of response breakdown and computers and pattern-seekers no longer employ the same "fixing" strategies. We shall consider each subgroup in turn.

B. Strategies of "computers" on statements in the form $a+b=a+b$, $a+b=b+a$, $a+b+c=a+b+c$ and $a+b+c=c+b+a$.

(1) First strategy. Statements in these forms systematically lead computers to mark them as "wrong" and to "fix" them. An excerpt from an interview indicates a common reaction to the item $5+3=3+5$:

Child: "That's impossible!"

Int.: "How come?"

Child: " $5+3$ isn't 3, it's 8."

Int.: "Can you fix it?"

(Child writes $5+3=8$)

Int.: "What about that 5 there?" (pointing to the second five in $=3+5$).

Child: "That's no good."

This is a very common "fixing" strategy of computers. They will employ the same strategy for a more complex statement such as $3+5+2=3+5+2$, "fixing" this as $3+5+2=10$.

(2) Second strategy. Some computers will consistently use another strategy: they perceive a statement in the form $a+b=b+a$ to be "two problems", one on the left of the equal sign, the other to the right, each requiring an "answer". In this case a variety of specific techniques are used to fix the statement. In the case of $5+3=3+5$, examples are:

(a) writing "two problems"

$5+3=8$

$3+5=8$

(b) "two answers" squeezed in the statement

$5+3=3+5$

8 8

(3) Third strategy. Some computers will rewrite the statement $5+3=3+5$ as $5+3+5=13$. Children using this "fixing" strategy would fix a more complex string, in the form $a+b+c=c+b+a$, as $a+b+c+b+a=d$, the second "c" consistently disappearing. When these children were interviewed this particularly transformation strategy became clear, at least from their point of view, and may be summarized as follows for the simpler case $a+b=b+a$ (the same argument also holds for $a+b=a+b$).

Given the statement $5+3=3+5$ in the interviews, children using this third strategy would explain that $5+3$ is not equal to 3. Therefore the statement is wrong. Since for computers the equal sign is interpreted as a separator/operator, the part of the string " $=3$ " is the portion that is "the wrong answer". However, rather

than dropping the second 5, as in the case of children using the first strategy, these children consider the second 5 to be "part of the problem". Hence the "problem" is $5+3+5$, and of course the "answer" is 13. The three in the segment " $=5+5$ " is simply "the wrong answer" and is dropped out. These children used the same argument in the case of the item $4+5=4+5$, which they "fix" as $4+5+5=14$, where again " $=4$ " is the "wrong answer" and drops out.

This fixing strategy is revealing of the tenacity of computers to interpret the equal sign as a separator/operator since only the number to the immediate right of the equal sign qualifies as the "answer" part of the statement, and other numbers further to the right are interpreted as "part of the problem".

There is a nuance that needs to be added to the strategy of computers concerning the forms $a+b=b+a$ and $a+b=a+b$. In one of the seven schools in which this study was conducted, one second grade teacher and one third grade teacher put considerable emphasis in their teaching on the commutative property of addition. In the interviews, the children performing as computers from these classes, would accept $5+3=3+5$ as correct. When asked "how come?" they would mechanically parrot "...because it's commutative". However, when presented with $4+5=4+5$, this was interpreted as "wrong" because "it's not commutative" and they would proceed to "fix" it as $4+5=5+4$ announcing "now, it's commutative". Their general performance on other forms of items leads us to suspect that their acceptance of items in the form $a+b=b+a$, and their transformation of items in the form $a+b=a+b$, is merely based on a mechanistic form-rule and does not indicate an interpretation of the equal sign other than as a separator/operator.

C. Strategies of "pattern-seekers".

It would perhaps be more accurate to speak of "pattern-seekers" as using "non-fixing" strategies as should become clear in a moment. Characteristic of pattern-seekers is their reliance on visual symmetries of form in a symbolic string. In the case of $5+3=3+5$, a typical pattern-seeker accepts this statement as correct. The justification given in the interview is to state, "well... There's a five there (on the left) and a five there (on the right), a three there and a three there, a plus there and a plus there... so it's alright". No computation is involved at all, merely the symmetry of elements on either side of the equal sign comes into play. A longer string as $3+5+2=3+5+2$ is accepted and justified on the same basis (there's a three there and a three there..., etc.).

However, pattern-seekers using a symmetry pattern to judge a statement fall into difficulties when presented with an incorrect

statement containing a subtraction, such as $8-5=5-8$. Pattern-seekers judge such statements to be "correct", using once again the argument "there's an 8 there and an 8 there, a 5 there, a minus there and a minus there... so it's alright". If the statement is judged to be correct, then of course from their point of view no transformation is required to "fix" the statement. In contrast a computer will state that this statement is "wrong", however, the justification is "because $8-5$ is equal to 3, not 5". The computer will "fix" the statement by using their typical "fixing" strategy, writing $8-5=3$ and, as in previous examples, the "-8" part of the string is just thrown out.

D. Commentary.

In the test form of presenting items to children and in the interviews we have used a greater variety of item forms than those presented here. Children have been questioned on identities ($8=8$), statements containing a subtraction operation in a variety of forms, and we have also presented some "unusual non-school" forms (such as $2+4=6-4+2$), that never seem to appear in standard school curricula. However, the discussion of the childrens' interpretation and fixing strategies on this more extended set of items, and the presentation of quantitative data, will need to wait until the publication of a more extended analysis.

However, even this brief qualitative presentation seems to provide additional confirmation for the existence of two quite different interpretations of the equal sign among first to fourth grade children: as a separator/operator and as a symbol of equality in terms of its reflexive property (see Erlwanger-Bélanger). The manner in which children, who hold the separator/operator interpretation (the computers), transform statements they consider wrong is consistent with this particular interpretation of the equal sign. On the other hand, the manner that pattern-seekers, who hold both interpretations, transform (or do not transform) statements which in fact are in need of "fixing" is again consistent with one interpretation or the other depending on the particular form of the statement.

While the interpretation of the equal sign as a separator/operator by children has long been identified in the research literature, the existence of the subgroups we have called "computers" and "pattern-seekers" adds another dimension to the attempt to probe childrens' struggles in understanding some aspects of the equivalence relation. An examination of childrens' transformation of statements they judged to be wrong, or the acceptance of incorrect statements, provides another glimpse into a matter of deceptive simplicity from the adult view, and of great complexity and difficulty from the perspective of the child. The adult understanding of childrens' understanding of the equivalence relation is still very incomplete.

A STUDY OF THE DEVELOPMENT OF NUMBER WORD SEQUENCE SKILLS
BY SCALOGRAM ANALYSIS

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RESUME

Une méthode d'analyse scolaire a été utilisée pour tester les propriétés d'échelle de 16 tâches appartenant à trois niveaux d'une théorie du développement des habiletés numériques (Fuson, Richards, et Briars, 1982). Quarante-deux enfants, âgés de 2 - 6 à 7 - 6, se sont vus attribuer des tâches touchant les niveaux suivants de la chaîne appartenant à la théorie de Fuson et al.: chaîne insécable (unbreakable chain), chaîne sécable (breakable chain) et chaîne dénombrable. Il a été ainsi découvert que ces tâches ne forment pas un ensemble pouvant correspondre aux propriétés d'une échelle, $I = .328$). Ces résultats suggèrent que la séquence de développement devrait être scrutée de plus près avant que les chercheurs ne se lancent dans une analyse des processus sous-jacents.

The concepts of developmental stage and developmental sequence are central to cognitive-developmental research. Although Piaget was one of the most enthusiastic stage and sequence theorists, he wrote that children's ability to count was independent of the operational structure of their thought (Piaget & Szeminska, 1952, p.61). However, many researchers have succeeded in using stages, levels, rules, and principles to describe the systematic nature of the acquisition of counting skills (Fuson et al., 1982; Gelman & Gallistel, 1978; Schaeffer et al., 1974; Siegler & Robinson, 1982).

One method for analyzing children's counting knowledge is the postulation of a developmental sequence which describes a series of stages, levels, or steps a child passes through on his way to greater expertise. At present there are many such sequences which describe various aspects of children's numerical understandings.

Given this availability of rich descriptions of stages, there has been a call by researchers to begin building more comprehensive and integrative models (Siegler & Robinson, 1982). The most popular method for building these models is to discover the processes which underlie the change from one level in a stage theory to the next (Bertenthal, 1981; Fuson et al., 1982).

Surely the attempt to move beyond static descriptions of stages toward an understanding of the processes which generate the stages is an important step for any developmental theory. However, before one can use a sequence to move from identifying stages to discovering processes it is absolutely necessary that the sequence be correctly specified. Unfortunately, researchers often make the step from sequence to process before the sequence has been adequately tested.

There are many definitions of sequence. An adequate test will depend upon the definition one chooses to use. But of the many criteria of sequences one property stands out as fundamental. That is, in order to use a sequence for the purpose of understanding developmental process it must be shown that the series of stages, levels, or steps are acquired in an invariant order. If one is to correctly account for the transitions between stages one must be certain that the stages are ordered correctly.

A direct test of the sequential order of acquisition of a set of skills requires the use of special methods. It is not enough to determine the average age at which children acquire certain skills or to test children on a set of tasks and then order those tasks on the basis of the percentage of children who pass at each age (Fischer & Bullock, 1981, p.8; Wohlwill, 1980 p. 346). The most obvious way to test a sequence for invariant order of acquisition is by a longitudinal study. However, other more practical methods are available. One alternative to longitudinal methods which provides much the same information is scalogram analysis (Fischer, Pipp, & Bullock, 1983; Green, 1956; Wohlwill, 1960). Given a set of tasks which are thought to represent a developmental sequence, an analysis of the patterns of successes and failures on these tasks by each child in a broad age range should reveal whether the tasks form a scalable set. If the set scales, then one may conclude that the ability to pass task n requires the ability to pass tasks 1 through $n-1$. Thus it may be concluded that the skills which determine success on the tasks are acquired in a sequential order.

The purpose of this paper is to apply the method of scalogram analysis to one of the very interesting sequences proposed to describe the acquisition of certain numerical skills. This sequence, proposed by Fuson and her co-workers (Fuson et al.,

1982; Fuson & Hall, 1983) postulates the existence of five levels in the elaboration of the number word sequence. At the string level, the number words can be produced but not heard as separate words. At the unbreakable chain level, the words can become separate objects of thought. At the breakable chain level, parts of the sequence can be produced beginning at arbitrary points. At the numerable chain level, the words become units which can be counted for purposes independent of sequence production. At the bidirectional chain level, the words can be produced easily and flexibly in either direction. Within each of these five levels are several steps which are postulated to emerge sequentially. If the tasks used to assess these levels form a scalable set then the next step for researchers would be to begin analyzing the processes which would account for the movement from one level to the next. However, if the tasks do not scale then more work is required on specifying the sequence before an analysis of processes is called for. In this paper, data will be presented that is relevant to a test of the unbreakable, breakable, and numerable chain levels of Fuson's theory.

METHOD

Subjects

Forty-two middle-class children from Denver served as subjects. They ranged in age from 2-6 to 7-6 with one boy and one girl from every 1/4 year age group. Two boys, one aged 3-6 and one aged 6-3 refused to complete the experiment.

Tasks and Procedures

Each child was brought into the experimental room by one or both parents and was invited to sit at a table with the experimenter and play some number games. Each child was presented with 16 counting and arithmetic tasks derived from Fuson et al.'s (1982) proposed developmental sequence. There were seven tasks from the

unbreakable chain level (Production of word sequence from one, Counting blocks with one-to-one correspondence, Cardinality rule, Make a set of size "a", Comes after, Comes before, Count-all solutions to addition problems), eight tasks from the breakable chain level (Count up from "a", Count-on solutions to addition problems, immediate production of Comes after and Comes before, Comes between, Count-up from "a" to "n", Count down from "b", and Count down from "b" to "a"), and one task from the numerable chain level (How many numbers between "a" and "n"?).

For the Production task the child was simply asked, "Can you count for Me?, I want you to count as high as you can." For the Counting blocks task the child was asked to count a row of 3 to 25 blocks. For the Cardinality rule task the child was asked to say how many were in a row of 3 to 25 blocks and asked to give $\frac{1}{3}$ -21 of them. For the Comes after and Comes before tasks the child was asked to say what comes before or after a certain number. If the child produced the correct answer immediately it was scored at the breakable chain level. If the child found it necessary to use a counting strategy the response was scored at the unbreakable chain level. For the Count-all task the child was presented with two separate sets of blocks and questioned "If there are $\frac{1}{2}$ blocks here and $\frac{1}{2}$ blocks there, how many blocks are there altogether?"

The procedures were similar for the breakable and numerable chain level tasks. For the Count up from "a" tasks the child was asked to start counting up from some number "a" ($a \neq 0, 1$). The Count-on solutions to addition task required the child to say how many dots there were altogether in two separate sets of dots after one of the sets had been covered with a card showing the numeral corresponding to the number of dots covered. Thus, the child was required to start counting at the numeral shown and arrive at the correct cardinality of the set. For the Comes-between task the child was asked to say what number came between two numbers "a" and "c" ($c-a=2$). For the Count-up from "a" to "n" task the child

was asked to start counting at "a" and stop when a/he got to "n". For the backward sequence tasks the child was asked to count down from "b" and to count down from "b" to "a". Finally, for the How many numbers task the child was asked to say how many numbers there are between "a" and "n" (e.g., "How many numbers are there between 3 and 7).

The tasks were presented in a random order except for the first two. Each session began with either the Production task, the Counting blocks task, the Cardinality rule task, or the Make a set task in order to increase the probability that the child would experience success on his or her first attempt.

In addition, each task was presented in at least three versions each at a different interval of the word sequence (1 to 5, 5 to 10, 10 to 30). Each session lasted from 15 to 40 minutes depending mostly upon the age and ability of the child.

RESULTS AND DISCUSSION

Green's index of consistency (a ratio of the difference between the observed reproducibility and the expected reproducibility to the maximum value of this difference) with $Rep_A = .875$ and $Rep_I = .814$, yields

$$I = \frac{.875 - .814}{1 - .814} = .328.$$

Following Green's recommendation that an I above .5 indicates satisfactory scalability one would conclude that the ordering of tasks postulated by Fuson and her colleagues (Fuson et al., 1982) do not form a scalable set.

Given the preliminary nature of these analyses and the small sample size one should not conclude that the tasks tested have no systematic relation to an invariant sequence of numerical skills. However, these results indicate that more work should be done before researchers move on to an analysis of the processes which

account for the transitions between levels within the theory. Namely, the ordinal nature of the sequence should be directly tested for. The use of scalogram analysis and broad variations in the sampling of age and task difficulty are particularly important to this end.

The present study points out certain roadblocks along the way to an explanation of the structures and processes in children's acquisition of counting skills. Future research might use an empirically derived ordering as a source of information for the construction of a sequence with better scale properties. If this were also done with task variations which cut across theoretical boundaries, patterns of data may appear which could be used to build more comprehensive theories.

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THE ACQUISITION OF MEANINGS FOR ARITHMETIC SYMBOLS

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RESUME

L'arithmétique, comme toutes mathématiques, est très symbolique. Une large part de l'enseignement de l'arithmétique doit donc être consacrée à aider les enfants à comprendre les symboles arithmétiques et leurs référents.

Les référents des symboles arithmétiques ne sont pas des représentations physiques ou visuelles mais des groupes d'opérations au sens d'actes manifestes, réels ou imaginés. Deux sortes d'arithmétique viennent de cette interprétation. L'arithmétique dite "opérationnelle" (c'est-à-dire, qui concerne les opérations) est une arithmétique composée d'expériences physiques avec des objets ou des images mentales. L'arithmétique "symbolique" reflète la précédente dans le sens qu'elle représente symboliquement les actes manifestes.

Il n'est évident qu'un enfant comprend un symbole arithmétique que lorsque la présentation de ce symbole fait venir à l'esprit de l'enfant le groupe d'opérations qui lui est rattaché. Des entrevues avec des enfants indiquent que le phénomène se produit rarement. Les stratégies pédagogiques qui s'appuient sur l'arithmétique opérationnelle semblent pourtant être très efficaces pour aider des enfants à apprendre le sens des symboles arithmétiques.

BACKGROUND

A child easily learns his native language symbol within a world of experience which constantly puts words together with things, ideas and actions. However, the same type of circumstances do not occur for a child learning the symbols of arithmetic. Though a child may experience natural occurrences of mathematical structure, these are rarely accompanied by their accepted mathematical verbalization or symbolization. Arithmetic symbolization is for the most part learned within a structured school setting. When learning the meaning of arithmetic symbols, the child necessarily must connect the arithmetic symbols with their appropriate referent so that the presentations of a symbol brings to mind the referent for that symbol.

As suggested in an earlier paper (Kolb and Clay, 1982), we believe the referents of arithmetic symbols to have a unique quality about them. They are operations, not in the sense of the operations of arithmetic but in the sense of overt acts real or imagined. For example, the operations which constitute the referent for " $1/3$ " could be those of defining a whole, dividing it into three equal parts and taking one of them. Thus, to assess a child's understanding of an arithmetic symbol, one must discover what the presentation of the symbol brings to the child's mind. Moreover, to teach a child the meaning of a symbol requires the teaching of the operations which in good usage match the symbol.

WHEN DOES A CHILD HAVE MEANING FOR AN ARITHMETIC SYMBOL?

Children seem to have a proclivity towards the generation of rules for working with and interpreting arithmetic symbols. The results of several clinical studies (Davis and McKnight, 1980; Peck and Jencks, 1981; Erlwanger, 1975) show that children often use their own rules for working with symbols and that the meanings children give to symbols often do not match the intended meanings of the teacher. Mathematical structure is not flexible enough to allow children to affix just any meaning to arithmetic symbols. A child

has meaning for an arithmetic symbol when the presentation of the symbol brings to the child's mind the referent for that symbol. Though the meaning is the possession of the child, it must also be correct in the sense that the appropriate operations are affixed to the symbol in the child's mind.

An example from some interviews with children may help to make the distinction more precise. Brian is a fourth grader. The following is an excerpt from an interview with Brian:

I: Do you know this symbol? (Writes $\frac{1}{3}$)

B: Yes, that's one-third.

I: What is one-third?

B: Well, I can show you one-third. (Draws )

I: (Draws ) Is this one-third?

B: No, because you didn't shade one of the parts.

I: (Draws ) Is this one-third?

B: Yeh, that's one-third.

I: Why?

B: Because it's one of three parts.

As the interview continues Brian was asked to draw other fractions. Always his pictures seemed to have equal divisions. However, when presented with pictures of unequal parts, he still okayed improperly drawn figures. Finally, we come to a crucial place in the interview.

I: I notice that when you draw your pictures you always make the parts equal.

B: Um...well, yeh.

I: Why?

B: Well, that's the way my teacher does it.

I: Why does she do it that way?

B: Well, I guess because it's easier or it looks better, but, .. um .. the parts don't have to be the same.

Obviously, Brian does have a set of operations for these fraction

symbols but not the set of operations which form the referent. Consider now an excerpt from an interview with Gerard, also a fourth grader.

I: Can you show me one-third?

G: Um, yep - like this. (Draws )

I: What about this (Draws ) , is that one-third?

G: Nope, because you could do this ( ; puts the dotted line in the figure). That's one-fourth.

I: Why can't it be one-third?

G: Because the parts are not the same size. The parts have to be the same.

Gerard does have meaning for the arithmetic symbol because he knows the referent operations of dividing the whole into equal parts and then taking one of them.

Erlwanger (1975, p.167) has indicated that interviews with children must necessarily be "flexible and exploratory" in order to discover their true conceptions. Elsewhere he indicates...

"...the extent to which teachers are capable of assessing children's understanding and thinking in mathematics and appraising the effectiveness of teaching strategies depends upon the teachers knowledge of the instruments and methods available and the kind of evidence they should look for. (1975, p.163). (emphasis ours).

Because enactive and iconic embodiments are in reality symbols themselves, they cannot be accepted as evidence of a child's meaning for an arithmetic symbol (Kolb and Clay, 1982, p.103). The true evidence for meaning exists only when a child presented with a symbol calls to mind the set of operations which form the referent for that symbol.

THE STUDY

Over a period of three months six fourth and fifth graders were interviewed weekly. Each session lasted approximately thirty to forty-five minutes and the sessions were described to the children as a time to talk math. The purposes of the study were (1) to discover what meanings the children possessed for some selected arithmetic symbols, (2) to define more carefully the true evidence of meaning for an arithmetic symbol, and (3) to explore the feasibility of instructional strategies which emphasized operational arithmetic. This report reflects preliminary analysis of the interview data collected.

In determining what meanings children had for a symbol, the results rarely disclosed evidence of operational thought. Rather, similar to results of other studies, children had meanings for symbols which could at best be described as aberrations of intended meanings. In the case of symbols such as $.7$, $1/5$, $3/8$ and 405 the children tended to explain what the symbol meant by showing iconic or enactive representations of the symbols. In other words, the presentation of a symbol typically brought to mind another symbol. In the case of symbols such as $407 + 394$, $1/5 + 3/4$ and $2/3 \times 1/4$ the children tended to explain what the symbols meant by offering rules for manipulating the symbols in order to get an answer. The symbols were the objects of the operations and were treated as if they were the referents.

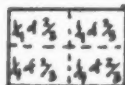
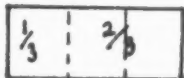
The probe for evidence of meaning proved somewhat frustrating at times for both the interviewer and the children. The children seemed to be firm in their belief that describing rules (correct or incorrect) for pushing symbols around indicated that they fully understood what they were doing. More probing questions tended to put them into the mode of trying to figure out what the interviewer was wanting them to say. The interviewer found it sometimes very difficult to refrain from teaching. As suggested by Erlwanger (1975)

every precaution was taken to refrain from making adult inferences from the children's responses. The only accepted evidence for meaning was an indication of operational thought.

If the true referents of arithmetic symbols are operations, then instructional strategies must emphasize operations as well as their encodings. Strategies used emphasized operations in either an enactive or iconic mode followed by recording activities followed by symbolic representations. For example multiplication of fractions was taught in the following way:

- (1) Instruct the child to cut with scissors a whole into three equal parts and push one aside.
- (2) Instruct the child to keep one-fourth of the remaining two parts and to give to the interviewer the other three-fourths.
- (3) Repeat with numerous other examples.
- (4) Instruct the child to label the parts as he goes.

e.g.



- (5) Instruct the child to compare "1/4 of 2/3" with the whole and label it.
- (6) Repeat these recording exercises with numerous other examples.
- (7) Give the child symbolic problems such as "1/5 of 3/4" and ask them to determine what part of a whole is being described.

Operational arithmetic experiences seemed effective in helping the children to learn referents. However, the long term effects of these strategies have not been determined. We hypothesize that instructional emphasis upon operational arithmetic will help a child to better understand arithmetic and lead to more instances of correct generalizations.

SUMMARY

For children to understand the arithmetic symbols with which they work, they must connect symbols with their true referents. Evidence that a child has meaning for an arithmetic symbol exists only when the presentation of a symbol brings to mind the appropriate referent operations. The child can acquire meanings for arithmetic symbols through carefully designed operational experiences later coupled with recording activities. Thus, the teacher can be influential in a child's personal construction of meaning and can by searching for evidence of operational thought more effectively assess a child's understanding of arithmetic.

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INTERPRETATIONS OF THE EQUAL SIGN AMONG ELEMENTARY SCHOOL CHILDREN

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RÉSUMÉ

Les résultats de recherches récentes indiquent qu'il existe deux interprétations du signe d'égalité chez les enfants de la 1^{ère} à la 4^e année du primaire. Chacune de ces interprétations semble influencer le comportement des enfants en ce qui a trait aux propositions mathématiques. De fait, les élèves peuvent être classés en deux sous-groupes selon l'interprétation qu'ils donnent au signe d'égalité.

La présentation sera centrée sur ces deux interprétations du signe d'égalité et sur le comportement mathématique des enfants.

Research by Behr, Erlwanger and Nichols (1975), Denmark (1976), and Van de Walle (1981) indicates children tend to view the equal sign more as an operator than as a symbol of equality. This operator view results in errors with equality statements such as $5 = 3 + 2$, $2 + 3 = 3 + 2$ and $4 + 5 = 6 + 3$. Kieran (1979) suggests that the operator view extends to the high school and probably affects students' ability to handle equations.

The above research has been based on small groups of children. How widespread is the operator view of the equal sign among elementary school pupils? Do pupils give any other interpretations to the equal sign? This paper discusses the results of a study conducted in five elementary schools.

A true/false 'Johnny' Test was administered to 527 grade 1-4 pupils, and 132 of these pupils were interviewed. Three types of items were used: Forward Problem-Answer (FPA), Backward Problem-Answer (BPA) and Arithmetic Identity (AI). A sample of grade 1-2 and grade 3-4 items are shown below.

Type	Grade 1-2 Item	Grade 3-4 Item
FPA: $a + b = c$	$6 + 4 = 2$	$80 + 60 = 20$
BPA: $c = a + b$	$5 = 3 + 2$	$52 = 30 + 22$
AI: $a = a$	$4 = 4, 3 = 5$	$83 = 83, 35 = 50$
$a + b = b + a$	$5 + 3 = 3 + 5$	$59 + 43 = 43 + 59$

A 'Johnny and his Math' technique was used in the study. The pupils were told Johnny was a 'real' boy who sometimes made mistakes in his work. They were being asked to help Johnny. In the Johnny Test they had to mark each item 'correct' or 'mistake', and then to 'fix' the item if it was wrong. In the interview the pupil was asked: What he thought of Johnny's item, what made him think so, and To 'fix' it if it was wrong. Later on the

pupil was shown $1 + 3 = 4$ (or $10 + 30 = 40$) and asked: What does the equal sign tell you here? The same question was asked about the line (bar) in the vertical form.

RESULTS

There are two interpretations of the equal sign among grade 1-4 pupils. One interpretation is associated with the Problem-Answer forms; the other with the Arithmetic Identity form. Typical responses to: "What does the equal sign tell you?" were:

"It tells you the example is finished so I put the answer."

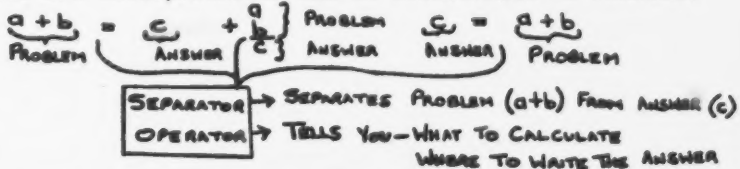
"The equal means: What's the answer?"

"We use it after a question where to put the answer."

"If we didn't put the equal sign in $1 + 3 = 4$, the teacher would think we did $1 + 34$."

"The line (bar) is like the equal sign on $10 + 30 = 40$. It just separates $10 + 30$ from the answer. So you always need an equal sign or the line if you are going down."

The S/O interpretation can be illustrated as follows:



Pupils interpret the 'equal sign' and the 'bar' as S/O signs based on their location between the problem and its answer. Thus they interpret $a + b = c$ as a Problem = Answer and $c = a + b$ as an Answer = Problem statement. In the interview pupils were able to decide if a statement was true or false. When a pupil said $8 - 6 = 2$ and $5 = 3 + 2$ were correct he meant the form and the answer were correct. When he said $6 + 4 = 2$ was wrong he meant the answer only was incorrect. The strategy for fixing it was to change the answer or the problem e.g. $6 + 4 = 2$ became $6 + 4 = 10$ or $6 - 4 = 2$. Many grade one and two

pupils however said $5 = 3 + 2$ was 'correct', but 'backwards' e.g. "Correct. It's backwards", or "It's backwards! It should be $2 + 3 = 5$ then it's right."

Only 4 out of 71 pupils rejected the backward forms. The average grade scores on these items on the Johnny Test are shown below.

GRADE SCORES

Grade 1-2 Items	1	2	3	4	Grade 3-4 Items
$6 + 4 = 2$	96	98	92	98	$80 + 60 = 20$
$8 - 6 = 2$	96	97	94	98	$82 - 60 = 22$
$5 = 3 + 2$	46	71	87	92	$52 = 30 + 22$
$2 = 6 - 8$	41	37	40	28	$22 = 60 - 82$

Grade 3-4 pupils obtained high scores on items 1 - 3. Grade 1-2 pupils obtained lower scores on $5 = 3 + 2$ because they marked it wrong. They rewrote it as $2 + 3 = 5$. Pupils who read $2 = 6 - 8$ as $8 - 6 = 2$ and said it was correct in the interview also marked it correct on the test. Hence the lower scores. The high score in grade 1 occurs since pupils marked the BPA form wrong and rewrote it as the FPA form. Thus the interview and Johnny Test results indicate that the S/O interpretation of the equal sign largely determines how pupils interpret Problem-Answer statements and the strategies they employ to deal with them. This interpretation enables pupils to handle Problem-Answer statements successfully; but it also leads to errors with backward forms such as $2 = 6 - 8$.

Pupils give two interpretations to the equal sign in Arithmetic Identities. One subgroup of pupils we call Computers continues to interpret the equal sign as a separator/operator (S/O). The second subgroup we call Pattern Seekers interprets it as a symbol of the equality relation based on the reflexive property of equality (E/R). The typical responses of Computers (C) and Pattern Seekers (PS) are shown below.

Items $4 = 4$ and $3 = 5$, or $83 = 83$ and $35 = 50$:

Grade 2 (C): "Well, $4 = 4$ doesn't make any sense because the equal sign always has to be at the end. You just put $4 + 4 = 8$ or $4 - 4 = 0$." (For $3 = 5$): "Well, I put $5 + 3 = 8$."

Grade 4 (C): "First al all this (=) should not be here. It doesn't make sense or anything. It should be $83 + 83 = \_\_$ or $83 - 83 = \_\_$." (For $35 = 50$): "It should be $35 - 50 = \_\_$ or $35 + 50 = \_\_$."

Grade 1 (PS): "It's very correct. Same thing." (For $3 = 5$):

"No, mistake. You put 3 here or 5 here." (writes $3 = 3$, $5 = 5$).

Grade 4 (PS): "It's right cause 83 does equal 83." (For $35 = 50$):

"No, that's wrong. It's $50 = 50$ and $35 = 35$."

Items $5 + 3 = 3 + 5$ and $59 + 43 = 43 + 59$:

Grade 2 (C): "Well, $5 + 3 = 3 + 5$ doesn't make sense. Who would ever say: $5 + 3 = 3 + 5$? Nobody would understand it."

Grade 4 (C): "Johnny miswrote it. He mixed it up, instead of putting the equal sign over here at the end."
(writes $59 + 43 + 59 = \_\_$).

Grade 1 (PS): "It's correct because 5 in front, plus in the middle and 3 at the end."

Grade 3 (PS): " $59 + 43 = 43 + 59$. Correct. They're both the same."

The interviews indicate that the S/O interpretation influences Computers to reject identities and rewrite them as problem-answer statements. To Computers, an identity does not have the correct form e.g. $4 = 4$ has no problem, and $5 + 3 = 3 + 5$ has a problem, but no answer. They explain this by saying the identity does not make sense because the equal sign is in the middle and not at the end. They also transform the identity into a problem-answer statement e.g. some of the strategies they use to correct $5 + 3 = 3 + 5$ are (1) put = at the end: $5 + 3 + 3 + 5 = 16$, (2) split $5 + 3 + 3 + 5$ into $5 + 3 = 8$, $3 + 5 = 8$ and (3) omit the wrong answer

3: $5 + 3 + 5 = 13$. The E/R interpretation influences Pattern Seekers to regard '=' as a balancing sign and to search for a number pattern on either side of it. The number pattern strategies for explaining why $5 + 3 = 3 + 5$ is correct are (1) use relational terms: " $5 + 3$ and $3 + 5$ are the exact same thing", or "they're both the same thing.", (2) reduce to $8 = 8$: " $5 + 3 = 8$, $3 + 5 = 8$ and $8 = 8$ ", and (3) use a number pattern demonstration: $5 + 3 = 3 + 5$.

These behaviours also appear on the Johnny test papers. Computers marked identities wrong and wrote down problem-answer statements as corrections so that their papers contained a great deal of written work and computation. The papers of Pattern Seekers contained very little written work. In fact, the test papers could be separated into a Computers pile and a Pattern Seekers pile. The average grade scores on identities in the Johnny test are shown below.

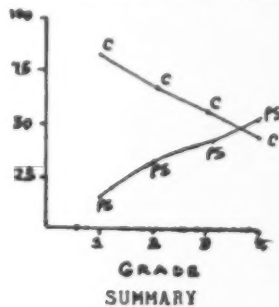
GRADE 1-2		AVERAGE % GRADE SCORES								GRADE 3-4	
ITEM	Form	1		2		3		4		ITEM	Form
		C	PS	C	PS	C	PS	C	PS		
$8 = 8$	I	42	70	34	82	50	89	52	80	$83 = 83$	I
$5 + 3 = 3 + 5$	II	5	20	34	83	26	81	33	97	$5 + 3 = 3 + 5$	II
$3 + 5 + 2 = 3 + 5 + 2$	III	21	72	30	78	35	91	41	92	$3 + 5 + 2 = 3 + 5 + 2$	III
$8 - 5 = 5 - 8$	IV	42	67	76	60	71	25	20	25	$2 + 3 + 5 = 5 + 3 + 2$	V

Computers in each grade obtain a much lower score than Pattern Seekers except on item 4. Pattern Seekers also used number pattern strategies in the interview to show that $8 - 5 = 5 - 8$ was correct. Therefore the interview and Johnny test results indicate that the S/O and E/R interpretations of the equal sign largely determine how

Computers and Pattern Seekers interpret identities and the strategies they employ to deal with them.

The above behaviour of Computers and Pattern Seekers in the interview and the Johnny test was consistent over a wide range of identities containing one and two equal signs regardless of their length or the size of the number. Computers rejected $2 + 4 = 6 = 4 + 2$ and $8 - 5 = 5 - 8$ in the same way as $4 = 4$. Pattern Seekers used the same number pattern strategies to show that they are true identities. In fact, the label 'Computer' describes a proficiency in computation and a tendency to behave like a computer programmed to detect and debug specific statement forms in ingenuous ways. The label 'Pattern Seeker' describes the tendency of consistently using number patterns in identities.

The percentage of Computers and Pattern Seekers among the 132 interviewees is illustrated graphically below.



There are two interpretations of the equal sign among grade 1-4 pupils: A S/O interpretation associated with the Problem-Answer Form and an E/R interpretation associated with the Arithmetic Identity form. Computers interpret the equal sign as a S/O only, while Pattern Seekers give both interpretations. Every pupil tends to

be either a Computer or a Pattern Seeker. The proportion of Computers is very high in grade one, but decreases gradually as one goes up the grades.

The results of the Computers are consistent with other research findings. The S/O and E/R interpretations together suggest that a pupil's understanding of equality is not an all-or-none process. It appears pupil's acquire the S/O interpretation as early as the first grade. Their acquisition of the E/R interpretation varies - some pupils seem to acquire it in the first grade and others after the fourth grade. A pupil's interpretation of the equal sign tends to determine his interpretation of statements and the strategies he employs to deal with them. Pattern Seekers can handle problem-answer statements and identities while Computers can only deal with the former. In fact the difficulties Computers have with identities arise from an attempt to handle them through a S/O interpretation of the equal sign.

The teaching and use of the equal sign as a relational symbol is often ill-defined because of the emphasis on computation and correct answers. However, there are hidden complexities behind the equal sign. The S/O interpretation seems adequate for computation, but it is narrow and misleading. Topics such as place value, decimals and fractions require a grasp of the reflexive property of equality. Therefore, the presence of Computers and Pattern Seekers in the same classroom should pose problems for both teachers and pupils.

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EARLY RELATIONSHIPS BETWEEN COUNTING AND CARDINALITY

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RESUME

La règle de cardinalité est l'habileté à dénombrer un ensemble et puis à répondre à la question "Combien y en a-t-il?" par le dernier mot prononcé lors du dénombrement. Trois différentes études dans l'application de la règle sur les premières performances de cardinalité ont été comparées: la reconnaissance immédiate de la cardinalité à partir de la configuration (subitizing) et le dénombrement de Schaeffer, Eggleston et Scott (1974), le dénombrement exact de Gelman et Gallistel (1978) et l'apprentissage de règles de Fuson et Hall (1983). L'utilisation de la règle de cardinalité ne dépendait pas de l'exactitude du dénombrement ou de la dimension de l'ensemble dénombré. L'emploi de la règle de cardinalité était équivalente pour de très petits ensembles (2,3,4), de petits (5,6,7) et de très grands ensembles (19,23,26) alors que l'exactitude du dénombrement diminuait de façon radicale de 75% à 47% à 0% pour les utilisateurs débutants de la règle de cardinalité. Parmi les utilisateurs originaux de la règle de cardinalité, 85% de ceux-ci avaient une reconnaissance immédiate de la cardinalité à partir de la configuration pour des ensembles de 2 objets, 30% pour des ensembles de 3 objets, et 4% pour des ensembles de 4 objets. Parmi les enfants qui n'utilisaient pas au prime abord la règle de cardinalité, 86% distinguaient et se rappelaient le dernier mot dit lors du dénombrement. Parmi ceux-ci, 85% montraient l'emploi de la règle de cardinalité après une session très courte d'enseignement. Les résultats ont confirmé l'étude de Fuson et Hall sur l'apprentissage de règles: la performance dans l'application de la règle de cardinalité est d'abord une chaîne simple de réponses dans laquelle la réponse à la question "combien?" ne se rapporte pas à l'ensemble des objets dénombrés c'est-à-dire ne fait aucune référence à la cardinalité. Les résultats confirment aussi une version modifiée de façon radicale de l'étude de Schaeffer, Eggleston et Scott sur la reconnaissance immédiate de la cardinalité à partir de la configuration et sur le dénombrement: quelques enfants utilisent d'abord la règle de cardinalité avec un ensemble dont ils peuvent reconnaître immédiatement la cardinalité à partir de la configuration et généraliser immédiatement la règle aux ensembles dont ils ne peuvent reconnaître la cardinalité immédiatement à partir de leur configuration. Les résultats n'ont pas confirmé l'étude sur le dénombrement exact.

Schaeffer, Eggleston, and Scott (1974) identified an early relationship between counting and cardinality which they termed "the cardinality rule": the knowledge that the last word said in counting some set tells how many entities there are in that set. They hypothesized that the cardinality rule is learned by children as they subitize (immediately apprehend the numerosity of) a very small set (2,3, or 4), count that same set, and then notice that the last number said in counting is the same as the subitized word for that set. Gelman and Gallistel (1978) found that the use of the cardinality rule (they used the term "cardinality principle") decreases markedly with increasing set size, as does accurate counting. They posited that children monitored their counting and when they were no longer accurate, they stopped using the cardinality principle. Fuson and Hall (1983) suggested that there are at least two levels of understanding involved in correct cardinality rule performance. The first is a simple chain of responses--count and then give the final count word when asked "How many?" This is simple rule learning. It involves a cardinal meaning only in that the number word is given as the answer to the "How many?" question. A higher level of understanding involves what Fuson and Hall termed a "count-cardinal transition" in word meaning: a transition from the counting meaning elicited when the word is said in counting (i.e., its meaning as referring to the object with which it is paired in the act of counting) to a cardinal meaning of the word when it is given as the response to the "How many?" question, i.e., here the referent of the word is a cardinal one--the word refers to the manyness of the set as a whole and is not just a referentless response to the "How many?" question.

Pergament and Fuson (1982) compared these three alternative accounts of beginning cardinality rule performance. They found that the use of the cardinality rule and accurate counting by 3- and 4-year-olds were independent for sets of 4,5, and 6, and use of the cardinality rule preceded accurate counting for sets of 9, 12, and 14. These findings were consistent with Fuson and Hall's

account of the cardinality rule, as their account is independent of counting accuracy and of set size, and were inconsistent with the Schaeffer et al. subitize and count account and with the Gelman and Gallistel monitoring count accuracy account. However, one study (Wilkinson, in press) contains some hint of support for modifications of both of the latter two accounts.

The present study sought to resolve the remaining ambiguities by assessing use of the cardinality rule and accuracy of counting for subitizable (2,3, and 4), nonsubitizable small (5,6,7) and very large (19,23,26) sets. A separate measure of subitizing of both linear and nonlinear sets (numerosities of 2,3, and 4) was also obtained for those children who did use the cardinality rule in order to assess the subitize and count hypothesis. The present study also explored the roles of memory in using the cardinality rule and undertook to ascertain whether children can be quickly and easily taught the cardinality rule.

METHOD

Subjects

The subjects for this study consisted of 42 middle-class 2- and 3-year-old children from a nursery school in Madison, Wisconsin. The 23 boys and 19 girls ranged in age from 2 years, 4 months to 3 years, 11 months.

Task and Materials

Cardinality rule task with objects. Each child was instructed to "count the xxxx (dolls, trucks, etc.)." given in a row. Correspondence errors made in counting were recorded using the error categories reported in Fuson and Mierkiewicz (1980). The child was then asked "How many xxxx are there?" This question was repeated. Three of the six trials were of subitizable sets (2,3,4), and three were of nonsubitizable small sets (5,6,7).

Cardinality rule task with stickers. This task differed from the cardinality rule task with objects only in that the stimuli consisted of stickers such as stars, animals, and smiling faces

affixed in a straight line to long strips of white cardboard and in that three sizes of sets were used: subitizable (2,3,4), nonsubitizable small (5,6,7), and very large (19,23,26) sets. Subitizing task. Twelve 3" x 5" cards were used in this task. Two cards each displayed one star, two stars, a line of three stars, three non-linear stars, a line of four stars, and four non-linear stars. The star cards were hidden behind a brightly colored postcard that served as a mask. A star card was shown above the postcard for a very brief period and then hidden again by the masking card. The child was asked to report how many stars there were.

Last word task. In each of the three trials in this task, the child selected the objects to count. While the child's eyes were hidden, the experimenter placed the (5,6, or 7) objects in a straight line on the counting tray. After the child counted the objects, the experimenter directed "Tell me what you said for this (experimenter point to the last object the child counted)."

Cardinality rule teaching. The cardinality rule teaching consisted of two verbal statements of the rule and three demonstrations of its use, as follows: "When you count, the last word you say tells you how many things there are. Watch me." (Put out 6 blocks) "1,2,3,4,5,6. Six. (said as experimenter counts blocks) Six. There are six blocks (experimenter gestures in a narrow ellipse over the set of six blocks)." "Watch again." Demonstration is repeated for 5 pigs and for 7 pennies, ending with the statement, "So the last word you say in counting tells you how many things you have."

Procedure

Children were tested individually in a small room in their nursery school. Testing was completed in anywhere from one to four sessions. Testing was interrupted when children refused to continue or were not attending to tasks. Testing resumed on a different day by repeating the last trial that had been given on the earlier day. All children were given a colored 3" by 5" file card of a color of their choice and were allowed to choose three

stickers to put on their card. This reward proved extremely effective in motivating children to return to complete the testing and to stick with testing on a given day. Total testing time ranged from 15 to 35 minutes.

RESULTS

Twenty children used the cardinality rule on two or more trials of the first cardinality rule task. Twelve of these used the cardinality rule from the first trial onward; eight seemed to discover the rule during the task, beginning to use it between the second and fifth trials. The response patterns of three of these eight "discoverers" fit a subitize and count pattern: they used the cardinality rule first on a trial of a set which they later subitized (on sets of 2, 2, and 3). However, all three children immediately generalized the cardinality rule—they used it on the very next trial (a nonsubitizable set) and on all trials thereafter. The response patterns of the remaining five "discoverers" did not fit a subitize and count account on either cardinality rule task: the first trial on which the cardinality rule was used was not a set subitizable by that child. These set sizes and the last words actually said by the child were 5 and 9, 6 and 5, 7 and 8, 4 and 8, and 4 and 4 (4 was not subitized on the subitizing task by either of the last two children). The results of the subitizing task for both the original cardinality rule users and the "discoverers" were similar: almost all children (85%) subitized two, only some children (30%) subitized three, and few children subitized four on at least 75% of the trials.

The initial cardinality rule users used the cardinality rule consistently across the set sizes, while counting accuracy went down with set size: from 78% to 56% to 0%. The "discoverers" did not use the cardinality rule completely consistently across the nine sticker cardinality rule trials. However, across all trials cardinality rule use did not vary with set size. As adjusted by a second chance at each set size, there were four, three, and five failures to use the cardinality rule on the very

small, small, and very large sets, respectively. Counting accuracy went from 71% to 33% to 0%.

Twenty-two children did not use the cardinality rule on the first cardinality rule task or used it on one early trial and then failed to use it again (three children). On the last word task, sixteen of these correctly gave the word they had said for the last object on one or more trials. Two additional children did so when the experimenter did the counting but could not afterwards do so when they themselves counted, and a third responded correctly to two of the three forced choice recognition questions. Thus 19 of these 22 children did demonstrate the ability to separate and to recall the last word said in counting. Three of these children then spontaneously used the cardinality rule on the next cardinality rule assessment. The remaining 13 who had demonstrated the ability to separate and to recall the last word said while they counted were given the cardinality rule teaching. Eight of them then used the cardinality rule on a mean of 6.8 of the 9 trials of the cardinality rule task with stickers (the cardinality rule was used on a range of from 4 to 9 trials). This mean and range is identical to the mean and range on this task of the eight "discoverers." Mean cardinality rule use was identical across the three set sizes. Three of the 13 taught children did not use the cardinality rule when they counted but did use it when the experimenter counted. The remaining two taught children did not use the cardinality rule when they or the experimenter counted; they had recalled the last word on only one trial of the last word task. Thus 11 of the 13 children taught did demonstrate use of the cardinality rule.

DISCUSSION

The results do not support the subitize and count account as originally proposed by Schaeffer et al. (1974). The use of the cardinality rule for sets of size 5 to 7 does not depend upon accurate counting, and of those 2- and 3-year-old children who use the cardinality rule, a few do not subitize any set (except

one), many do not subitize sets of 3, and almost all do not subitize sets of 4. Furthermore, not all children follow any kind of subitize and count approach. However, the results do support the following radically revised version of the subitize and count description: some children first use the cardinality rule on a set which they can subitize and then immediately (on the very next trial) generalize the rule to sets which they cannot subitize. The main points of revision are underlined. First, the subitize and count account applies only to some and not to all children. Second, the results of the subitizing task indicate that one cannot characterize sets of a particular size as subitizable but only as subitizable by a particular child (and this must be determined for each child). Third, it does not seem necessary for a child to have many separate subitize and count experiences before discovering the cardinality rule. Rather one experience may be (and for the children in this study, was) sufficient. This implies that the subitize and count approach is really much more closely linked to the rule-learning approach--one subitize and count experience may trigger the rule, which is then applied immediately to larger sets.

The evidence on count accuracy and cardinality rule use indicates that it is not necessary to count accurately when first discovering or using the cardinality rule (only one of the five "discoverers" counted accurately on their first cardinality rule use trial), when generalizing the rule to small sets (5, 6, and 7), or when generalizing it to very large sets (19, 23, 26). One child (a subitize and count "discoverer") did follow the modified Gelman and Gallistel pattern of use of the cardinality rule on sets 2 through 7 and not on the very large sets, but otherwise there was no support for this position.

The Fuson and Hall rule-learning account was supported by several kinds of data: more than half of the "discoverers" spontaneously discovered the cardinality rule on a nonsubitizable set, some children given the last word task then spontaneously demonstrated

the cardinality rule (i.e., focusing attention on the last word was sufficient for the discovery of the rule), and all of the children who could consistently separate and recall the last word they said in counting were easily taught the cardinality rule. This account raises the question: "In the normal course of events, what might lead to a child's discovery of the cardinality rule?" One part of an answer rests in the nature of the cardinality rule situation itself. When a child is counting, the most salient word is the last word said in counting because it is not surrounded on two sides by a distracting word. Another part of the answer comes from the responses to the forced choice questions; these demonstrated that the children who did not use the cardinality rule had a strong response bias (73% of the responses) towards answering a question with the last of the three alternatives named. Thus when a child is ready to hazard a guess to the cardinality rule question, both of these factors would seem to lead to an accurate guess--the last word said by the child in counting.

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THE DEVELOPMENT OF NUMBER WORD SEQUENCE SKILLS

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RESUME

Fuson, Richards et Briars (1982) ont émis l'hypothèse que l'apprentissage de la suite usuelle des nombres par les enfants amène ces derniers à une représentation de plus en plus élaborée de cette suite. Ils ont distingué des niveaux de développement dans cette élaboration, niveaux constitués d'habiletés et de relations de complexité croissante dans l'utilisation de la comptine des nombres.

Le but de cette étude était de fournir des données touchant plusieurs des premières habiletés et relations identifiées par Fuson et al. Soixante enfants de classe moyenne, âgés de trois ans et demi à six ans ont servi de sujets à raison de douze enfants par tranche d'âge de six mois. La production de séquences à rebours arrivait environ deux ans après celle de séquences vers l'avant. Le comptage à partir de un jusqu'à un nombre donné a était significativement plus facile que le comptage en avant à partir de ce nombre a. L'hypothèse de l'existence des niveaux de développement dits de la chaîne insécable (Unbreakable chain) et de la chaîne sécable (Breakable chain) a été soutenue et la transition vers le ~~niveau de la chaîne sécable~~ pour les mots référant aux nombres inférieurs à dix semblait accomplie autour de quatre ans chez beaucoup d'enfants. La séquence de comptage en avant à partir de a était bien en place vers quatre ans et demi alors qu'un bon nombre d'enfants de quatre ans et demi à cinq ans en étaient encore à l'apprentissage de la relation Et-Alors pour les mots référant aux nombres plus petits que dix. Les enfants commençaient déjà à élaborer la première partie de la suite (pouvant, par exemple, compter jusqu'à a pour des mots comme huit et dix) alors qu'ils en étaient encore à apprendre les mots de la comptine pour des nombres plus grands (par exemple, le mot seize).

Fuson, Richards, and Briars (1982) proposed that after children learn the standard sequence of number words, the representation of this sequence becomes increasingly elaborated and children become able to demonstrate increasingly complex skills using the number word sequence and come to construct various relations on the words in the sequence. Data concerning the earliest skills and relations were sparse in the Fuson et al. study. In particular there were no data from the same children across all the earliest skills and relations. The purpose of the present study was to provide such data.

METHOD

Subjects

The subjects were 60 middle-class children aged 3-6 (3 years 6 months) to 5-11, twelve children in each half year age group. The children all attended a pre-school/kindergarten in a northern suburb of Chicago.

Tasks

The five sequence tasks discussed in this paper were a subset of tasks given in two thirty-minute sessions.

Forward sequence task. The child was asked to "count as high as you can." Whenever a child stopped producing words, a prompt was given: "Go on." or "Can you count higher?". Children who reached 121 were stopped there. This task was given three times: once at the end of the first session, in the second session at the end of the other word sequence skills, and after the object counting trials of the second session.

Backward sequence task. The experimenter first modeled producing a backwards sequence from three (i.e., "3,2,1"), and the child was asked to do likewise. The child was then asked to count backwards from 30 (or from the child's highest forward sequence word if this was less than 30). If a child failed to count backwards accurately from 30, the child was asked to count backwards from lower numbers (given in order): 29, 20, 19, 10, 9, 5. Any child who counted backwards from 30 was asked to count backwards from 70. This backwards task was given after the three

sequence tasks below so that backwards sequence production would not interfere with those tasks.

Counting up from one to a. Three trials were given. The child was asked to count up to 8, to 10, and to 16. This task was the first task given in the second session.

And-Then relation task. The word which comes immediately after a word is in the And-Then sequence relation to it: e.g., 6 And-Then ____ is answered by 7. In this task the experimenter said, "When you count and say 3, 4 comes right after, like 3,4 (stress-ing the count). When you count, what number comes right after 3?" Wrong answers were corrected and the explanation repeated. Six trials of the question, "When you count, what number comes right after ____?" were given. Three trials were between 6 and 9, and three were between 16 and 19.

Counting up from a. The experimenter asked the child to start counting from 3 and demonstrated such counting (3, 4, 5, 6) if the child did not do so. Six trials to the direction "Start counting from ____" were then given. Three trials were between 6 and 9, and three were between 16 and 19. This task and the And-Then relation task were counterbalanced. These tasks were the second and third tasks given in the second session.

RESULTS AND DISCUSSION

Means by age groups for the forward sequence and backward sequence tasks are given in Table 1. One-way analyses of variance revealed main effects of age on both tasks, $F(4,59) = 10.36$ and 7.98 , respectively. Solid double lines in Table 1 indicate the results of Newman-Keuls contrasts between age groups. For the forward sequences, there were significant increases in the mean score for all contiguous age groups except from the old four-year-olds to the young five-year-olds. These data, as did those of Fuson et al., show a considerable extension of the number word sequence by 4½-year-olds, who are beginning to learn the decade structure of the sequence. Many of the 5½-year-olds in the present study had completed that task, due partly to the presence of a teacher who had children practice counting at various times.

Table 1
Forward Sequence and Backward Sequence Means
and Standard Deviations by Age

Age	Forward Sequence ^a	Backward Sequence ^b
	X SD	X SD
3½ to 4	<u>15</u> (6)	<u>2</u> (3)
4 to 4½	<u>23</u> (13)	7 (4)
4½ to 5	46 (39)	10 (8)
5 to 5½	<u>49</u> (37)	<u>10</u> (9)
5½ to 6	86 (37)	18 (10)

- a. The highest word reached accurately on 100% of the forward sequence trials. The age means for the highest word in the single best trial were quite similar: 15, 24, 50, 52, and 87, respectively.
- b. Only one child, a young four-year-old, could count backwards from 70; that score was entered here as the ceiling score of 30.

For the backward sequences, there was a significant improvement from the old three-year-olds to the young-four-year-olds and then no further improvement until age 5½. The ability to produce a backwards sequence lags very considerably behind the ability to produce a standard forward sequence. Our data are consistent with those reported by Fuson et al.: more than half of the five-year-olds are unable to produce a backwards sequence above ten. This difficulty in the mere production of backwards sequences does not support a very optimistic viewpoint concerning the ease with which all children will be able to use for subtraction problems the backwards sequence strategies specified by Fuson et al. to occur at the later elaborative levels of the sequence: counting down to counting down from. Note also, as in the Fuson et al. data, the existence of very large standard

deviations in both sequence tasks; these indicate the presence of huge within age variability in sequence production abilities.

Counting up to a was described by Fuson et al. as requiring the ability to remember the word up to which one was counting while one was producing a forward sequence plus the ability to check where one was in the sequence with the remembered number. This analysis was supported by the nature of the errors made by our subjects on this task: on almost every trial on which a child could produce a forward sequence up to a (as measured on the forward sequence task), the error consisted of continuing to count past the specified number a, i.e., the children were not able to maintain a in memory or to check a with their current word while producing a forward sequence. At the three youngest age groups, scores on counting up to the words eight and ten were significantly higher than those on counting up to sixteen, $t = 2.97, 1.77, 2.60, df = 11, p < .05$, means of 38% vs. 71%, 54% vs. 88%, and 67% vs. 100%, respectively. The lower means for counting up to sixteen reflected both those children who continued counting past sixteen when asked to count up to sixteen and those whose forward sequences did not extend to sixteen. Most (13 out of 16) of the latter children did demonstrate the sequence skill "count up to a" for the words eight and ten, thus supporting a major aspect of the Fuson et al. analysis: a later portion of the number word sequence may be being learned (e.g., the word "sixteen") while an earlier portion of the sequence is already being elaborated.

Counting up to a was hypothesized by Fuson et al. to occur at the Unbreakable Chain level in which the child could not yet break into the number word chain and begin counting (i.e., could not count up from a). This position was supported by our comparison of performance on counting up to a with counting up from a. Scores on two comparable trials from each task (8 and 10 vs. 7 and 9) were used. At the three youngest age groups, counting up to a a specified number was significantly easier than starting to

count up from a similar specified number, $t = 3.92, 1.77$, and 1.77 , $df = 11$, $p < .05$ with means of 71% vs. 13%, 88% vs. 71%, and 100% vs. 84%. Performance on both tasks approached ceiling for the two oldest groups. The means of 71% and 13% for the youngest age group (age 3½ to 4) indicate that counting up to a is already a fairly well executed skill but practically no child can break his number word chain and start counting up from an equivalent word in that chain. Thus existence of the developmental levels of Unbreakable Chain and Breakable Chain in the elaboration of the number word sequence was supported, and the transition to the Breakable Chain level for words below ten is seen to be at about age four.

Fuson, Richards, and Briars discussed two possible ways in which children might respond to questions about the And-Then relation on a word: they might count up from one to that specified word (i.e., might use the sequence skill 'count up to a') or they might start counting at that specified word and produce the next word (i.e., might use the sequence skill 'count up from a'). In the present study, ten of the 60 children gave on 25 of the 360 trials behavioral evidence of using count up to a or paused long enough before answering to have counted up to a. Use of this skill was not consistently helpful: only 44% of these trials resulted in the correct answer. Five children gave direct evidence that they were using the count up from a skill: e.g., they would say "6,7" when asked what comes after 6. However, many of the children could have been using this skill without saying the first word out loud--the ratio of those with higher scores on the count up from a task than on the And-Then relation task compared to those with higher And-Then scores was 19 to 7 for words below ten and 21 to 4 for words between ten and twenty. This pattern is what one might expect if the count up from a skill is being used to answer And-Then questions. A third possibility is that an And-Then question is not answered by thinking of the number word sequence at all but rather is answered by thinking of some word associated with the given word

in semantic memory. One would expect that the word immediately following a given word would become a strong associate of it, especially in young children who have few number word experiences other than counting for words other than very small numbers. This approach would be expected to produce some correct answers. It might also produce some incorrect answers, such as the word immediately before a given word, which might also be expected to be a strong associate of the given word. In fact, a substantial proportion (32%) of all given wrong numbers (as opposed to "I don't know", etc., responses) in the And-Then task were the word immediately preceding the given word. Such words coming immediately before were not produced by a consistent misunderstanding of the task, for no child consistently produced such errors. Our data thus indicate that children may use at least three different methods of generating the And-Then relation.

Table 2 presents the performance by age and by size of number word on the count up from a sequence skill and the And-Then relation task. One-way analyses of variance by age on each task for each size of set indicated significant age differences for both sizes of words for both tasks, $F(4,59)$ ranging from 6.05 to 14.8, $p < .002$. Newman-Keuls analyses indicated that performance on the count up from a task increased significantly between the three-year-olds and the young four-year-olds for both sizes of number words and that performance increased significantly on the And-Then task between the old three-year-olds and young four-year-olds and between the latter and the old four-year-olds for words between six and ten and only between the two youngest groups for words between 16 and 19. The count up from a sequence skill within the sequence as a Breakable Chain thus seems to be well established by age $4\frac{1}{2}$, while the And-Then relation for words below ten is still being learned by a good portion of older children ($4\frac{1}{2}$ to 5) and for words between ten and twenty is learned more gradually from age $4\frac{1}{2}$ to 6. Correlations within each age group between performance on these two tasks were always

considerably higher for words between 16 and 19 (correlations ranged from .60 to .87) than for words between 6 and 9 (correlations from -.43 to .52). This may indicate a difference in the way the And-Then relations are derived on words below and above ten.

Table 2
Per Cent Correct by Size and Age
on Count Up From a and And-Then Relation Tasks

Task	Age				
	3½-4	4-4½	4½-5	5-5½	5½-6
Count Up					
From <u>a</u>					
<u>a</u> < 10	22	72	81	86	100
10 < <u>a</u> < 20	17	50	67	67	89
And-Then					
<u>a</u> < 10	14	53	78	83	92
10 < <u>a</u> < 20	0	36	58	64	75

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DETERMINANTS OF YOUNG CHILDREN'S PROBLEM SOLVING BEHAVIOR

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RESUME

Au cours des dernières années, un grand nombre de recherches ont été centrées sur les procédés utilisés par de jeunes enfants au cours de la résolution de problèmes verbaux en addition et en soustraction. On a trouvé que la plupart des procédures appropriées de résolution de problèmes étaient basées sur le comptage. Les conditions sous-jacentes au choix de stratégies de solution par les enfants sont les suivantes: les liens sémantiques entre l'énoncé du problème et la procédure de solution, le niveau d'abstraction de la procédure mathématique et de la connaissance sémantique, la situation expérimentale, l'efficacité du comptage et la forme préférée de comptage, et le comportement idiosyncratique du sujet.

Recent years have seen an increase in the interest in processes used by children in problem solving, rather than in simply whether problems are correctly solved or not. One area in which this interest has been very strong is in the domain of addition and subtraction verbal problems (e.g. Carpenter, Moser, & Romberg, 1982; Nesher & Teubal, 1975; Riley, Greene, & Heller, 1983). The purpose of this presentation is to set forth some hypotheses that explain the underlying causes for the procedural behavior of children that has been observed by a number of researchers.

Two teams of investigators (Briars and Larkin, 1982; Riley, et al, 1983) have proposed computer models that attempt to simulate children's problem solving procedures with respect to certain classes of arithmetic verbal problems. While the authors freely admit that their efforts should not be accepted as final products, both simulations appear to suffer from the basic assumption that, at a particular period in time, children's behavior is relatively stable and predictable. However, data from a three-year longitudinal study conducted by Carpenter and Moser (1982) suggest that this assumption is very tenuous. As might well be expected from the human condition, there was a large amount of variability of behavior between subjects of the study. But more importantly, the variability of behavior within individual subjects was equally as extensive. This variability leads to the thesis that there are a number of possible different determinants of pupil behavior at play. We shall consider these possibilities in what follows. But, first some brief background.

BACKGROUND

Simple one-step verbal problems in addition and subtraction may be categorized according to their semantic structure. One class is called the Cause-change in which some initial state is subjected to a transformation, resulting in a final state. The explicitly described change suggests a temporal action and this change or transformation can be positive or negative. Since either the initial

state, transformation, or final state may be the unknown, their class admits to six types of problems. The second class of problems, called Combination problems, describe a static relationship existing between a whole entity and two disjoint sub-entities that make up the whole. This class, sometimes described as Part-whole problems, admits only two problem types, one when the whole is unknown and the other when one of the parts is unknown. Comparison problems comprise the third class of verbal problems. Here, the static relationship concerns itself with the comparison of two disjoint sets, one called the compared set and the other the referent set, and the numerical difference between the measure of the two sets. Since the direction of the comparison can be positive or negative (how much larger?; how much smaller?) and since any one of the three elements (compared set, referent set, difference) may be unknown, this class also admits to six different problem types.

Some of the problem types are solvable by procedures which can be described as corresponding to the operation of addition and others by subtraction procedures. With young children, these procedures are counting based, at first with objects and then later without objects. As the children progress in school, much of the counting gives way to use of number facts and computational algorithms. Addition counting procedures with objects, such as blocks or fingers, essentially involve the construction of two sets, the formation of the union of those sets and then the enumeration of that union set. Several distinct procedures with objects have been observed for subtraction, separating a subset from an initially constructed set followed by the enumeration of the remainder set; incrementing of an initially constructed set until a specified size is reached followed by the enumeration of the set that had been added on; and the comparison, usually by means of a one-to-one correspondence, of two previously constructed sets followed by the enumeration of the excess of the larger over the smaller. These three procedures can be described as subtractive, additive, and comparative, respectively.

Without objects, four counting sequence procedures are possible, all of which are identified by the recitation of a memorized string of number words beginning with a word other than "one". Two are forward and two are backward, and each of these is subdivided into two parts depending upon whether the number of counting words to be spoken is known in advance (in which case the final word spoken is the answer) or whether the final word to be spoken is known in advance (in which case the number of words spoken is the answer). These all are abstract analogues of certain counting procedures used with objects, though we note there is no counting sequence analogue to the comparison (one-to-one correspondance) procedure.

WHAT MAKES CHILDREN BEHAVE THE WAY THEY DO?

In what follows I distinguish between procedures and strategies. A procedure is simply a way of behaving; some have been briefly described above. In contrast, a strategy is thought of as a plan of action, more or less consciously worked out in advance. In most instances using a strategy involves making a choice among several available procedures.

Direct Modeling. This is a procedure that emerges at an early stage. Upon hearing or reading a verbal problem, the child uses physical objects to represent the entities of the problem and actions upon these objects to represent the actions described in the problem. As such this procedure is limited to dynamic problems of the Cause-change category. The modeling behavior tends to be very sequential in nature and must begin with the modeling of the initial set. As such, problems where the initial states are unknown are unsolvable. These procedures will vary according to problem type and thus are semantically linked to problem statements. A child can solve several different types of problems, each by a different procedure. While, on the surface, the child exhibits a somewhat rich repertoire of problem-solving behavior, in fact he/she is operating at a fairly primitive level and cannot be presumed to have the ability to choose among the various behaviors that have

been exhibited. Of course, the child cannot use direct modeling if he/she lacks the linguistic and semantic knowledge to understand the verbal problem in the first place.

The "Action" Schema. Correctness data (Carpenter & Moser, 1982; Riley et al, 1983) on addition/subtraction verbal problems as well as research in areas such as geometry and interpretation of the equal sign indicate that children tend to interpret situations in a dynamic fashion. Thus, it is hypothesized that children impose action upon static solutions such as combine and compare problems and then attempt to solve using methods appropriate to a dynamic situation.

Abstraction of Counting Procedures. At the strictly procedural level, children abstract the essence of counting and are able to substitute a count-on procedure for counting-all with objects, for example, or use a counting down procedure instead of a separating behavior with blocks. It is quite possible that a child may advance in abstraction in procedural terms and yet still be semantically bound to problem texts. In other words, a child might be able to count up (the abstract analogue of adding-on with objects) but not yet see that procedure as a possible solution method for a change/separate problem.

Equivalence of Procedures. This, too, is evidence of abstraction on the child's part but on a separate dimension which might be described as mathematical. Either at the counting with objects level or at the more abstract counting sequence level, the child is aware of the equivalence of procedures in the sense that two procedures can give the right answer. For example, the count on from smaller (MAX) procedure is seen to be equivalent to the count on from larger (MIN) one. If a child sees that additive counting is equivalent to subtractive counting then a strategy could be seen to emerge where the child was then free to choose between them for any problem for which either procedure might be viewed as semantically

appropriate.

The Part Whole Schema. Resneck (1983) and others have suggested that children come to have a so-called part-whole schema which I see as abstraction along still a third dimension which is semantic in nature. Having this schema, a child is able to interpret all simple one-step problems of addition/subtraction as describing a static relation between a whole and its two distinct sub-sets which are its parts. The whole is composed of the two parts and in turn, these two parts constitute the whole. These two aspects (whole-part and part-whole) may emerge at separate points in time. With this schema, the child is able to interpret and solve more difficult problems such as initial-state-unknown change problems.

The Choice Strategy. Armed with the ability to abstract along any or all three dimensions mentioned above, the child can thus be presumed to have a strategy which permits a choice from among several alternative procedures.

- 1) Efficiency. A child chooses a counting procedure which requires the least number of steps. Thus, count-on from larger (MIN) would be preferred over count-on from smaller (MAX). In subtraction, $9 - 2$ a child might choose backward counting, whereas for $9 - 7$ the preference would be for forward counting.
- 2) Counting Preference. As an alternative to efficiency, a child might choose a counting procedure simply because he/she is more comfortable with it. Most evidence suggests there is a strong preference for forward counting, regardless of problem data.
- 3) Interview Conditions. A child may choose a procedure for reasons independent of semantic, syntactical, or mathematical reasons. For example, a child might use objects

to count with even though capable of more abstract procedures simply because the objects were there and the child assumed an interviewer expected them to be used. Fatigue due to late-in-the-day interview could cause different procedures to be chosen. Often, the preceding problem has an effect on behavior.

- 4) Number Size. This effect has been well documented. Certain more abstract procedures are only available for numbers less than 10.
- 5) Syntactic Variables. Of particular notice is the order in which the numbers are presented. Earlier we noted the dynamic instincts of young children. One aspect of this is to process problem data in the sequential order in which it is presented. This can, in turn, affect the choice of procedure.
- 6) Idiosyncratic Behavior. Often children will choose a procedure for no more apparent reason than "I just felt like it."

Interactions between the various factors presented here will be discussed during the presentation.

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EARLY MULTIPLICATION AND DIVISION

Leslie P. Steffe, Paul Cobb

RESUME

La construction, par les enfants, de concepts de multiplication et de division requiert l'abstraction complète de tout contexte particulier. Il devient de plus en plus clair qu'il s'agit là, pour les enfants, d'une abstraction réflexive très exigeante. Même lorsque les enfants arrivent à faire cette abstraction, leurs procédures de multiplication et de division sont souvent sans relation avec leurs concepts. La notion de schème est utilisée pour unifier, sur le plan théorique, concepts et procédures. Nous décrivons les premiers concepts de multiplication et de division et indiquons comment ces concepts impliquent des procédures. Notre principale prémisse est que les schèmes itératifs qui coordonnent ces premiers concepts et procédures peuvent servir de matériel de base pour des abstractions plus poussées. La spécification de ces schèmes et du progrès de leur développement peut s'avérer un guide précieux pour l'éducateur dont le but est d'encourager ce développement au moyen de méthodes d'enseignement appropriées.

EARLY MULTIPLICATION AND DIVISION¹

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The real problem which confronts mathematics teaching is not that of rigor, but the problem of the development of 'meaning', of the 'existence' of mathematical objects (Thom, 1973, p. 202)

The problem of meaning in arithmetic has been recently examined from the constructivist perspective (Cobb, 1983; Thompson, 1982; von Glasersfeld, 1980). The meaning of a symbol like "4" consists of the child's interpretation of it based on the schemes that are available. The primary task of the child in the development of meaning in arithmetic is viewed as the construction of such schemes (or mental activity). For Piaget (1980), "All action that is repeatable or generalized through application to new objects engenders by this very fact a 'scheme'" (p. 124). The realization that schemes can function at different levels of abstraction makes it plausible to think of mathematical objects as the phenomenological correlates of schemes. von Glasersfeld's

(1980) recent characterization of a scheme lends credibility to this point of view. "'Schemes' are basic sequences of events that consist of three parts. An initial part that serves as trigger or occasion. In schemes of action, this roughly corresponds to what behaviorists would call "stimulus", i.e., a sensory-motor pattern. The second part, that follows upon it, is an action ("response") or an operation (conceptual or internalized activity). . . the third part . . . is what I call the result or sequel of the activity . . . " (p. 81). This structural analysis of scheme draws from Piaget's characterization of scheme as generalized action patterns. Placing emphasis on "situations" that serve as trigger of the scheme should not be interpreted as these situations being external to the individual. The crucial difference between the constructivist and behaviorist views of the stimulus or trigger has been expressed by Piaget (1964).

A stimulus is really a stimulus only when it is assimilated into a structure and it is this structure which sets off the response. Consequently, it is not an exaggeration to say that the response is there first, or if you wish, at the beginning there is structure. (p. 15)

The meaning of a product such as 6×4 is determined by the schemes into which it is assimilated. In general, arithmetical knowledge is viewed as the coordinated schemes of actions and operations that a child has constructed at a particular point in time. Focusing on children's cognitive schemes opens up exciting new vistas for mathematics instruction in the elementary school

Children can be enormously powerful when instruction is in harmony with their schemes. Children's generative power is demonstrated by Tyrone, a seven-year-old boy, solving " 4×6 " by counting.

He said "6, 12, 18, 24." Later, he was presented with a problem he asked to do: " 20×20 ". Tyrone sequentially put up fingers twenty times in synchrony with saying "20, 40, 60, 80, . . . , 340, 380, 400". He put up each of his ten fingers ("20, 40, . . . , 200") and then put up his ten fingers again ("220, 240, . . . , 400"). He stopped counting without suggestion, indicating that he fully intended to repeat 20 twenty times. The next task given was " 30×20 ". He began counting from "four hundred -- "420, 440, 460, 480, 500, 520, 540, 560, 580, 600!" He again put up fingers sequentially and stopped counting on his own. (Steffe, 1982. p. 4)

From the observer's perspective, Tyrone's behavior can be viewed as indicating a multiplicative scheme, where the first part of the scheme consisted of the meaning of the expressions ("4x6" or "20x20"). These expressions triggered what followed -- i.e., counting by sixes or by twenties. The result, or sequel, of the counting activity was indicated by his ability to start where he left off -- at "400". This external behavioral analysis is offered only to illustrate that children's multiplicative behavior can be interpreted as schemes. It is not meant to be a model of the internal mental events that might have transpired either before, during, or after counting.

Multiplicative and Divisional Schemes

Concepts and procedures

The first part of a multiplicative or divisional scheme consists of the child's conception of situations in which multiplicative or divisional schemes are used. The second part consists of the child's current processes or procedures (the algorithms) of multiplying or dividing, and the third part consists of a review of the procedures, pulling from them the results. In constructivism,

"Concept" refers to any structure that has been abstracted from the process of experiential construction as recurrently usable, for instance, for the purpose of relating or classifying experiential situations. To be called "concept" these constructs must be stable enough to be re-presented in the absence of perceptual input (von Glasersfeld, 1982, p. 194).

A concept can, on the one hand, be implemented and thus imply procedures and, on the other hand, it can be informed by procedures. In other words, the child structures experience in terms of his or her concepts. Guided by current purposes or intentions, the child performs a procedure by filling out, or expressing, this structure. In Piaget's (1964) words, "the response is there first, or . . . at the beginning there is structure" (p. 15). This becomes apparent as soon as one considers how children use procedures purposefully to achieve their goals. Tyrone, for example, intentionally counted by 20's to solve 20x20. He anticipated that he could solve his task by iterating a unit of 20; he knew what he was going to do. Concepts are anticipatory structures which guide the use of procedures within a goal-directed

framework. In general, "anticipation is nothing other than . . . application of the schema (or scheme) . . . to a new situation before it actually happens" (Piaget, 1971, p. 195).

An Elementary Divisional Scheme

In order to illustrate the sense in which concepts and procedures are not artificially separated in our theory, we present and analyze and elementary divisional scheme. In the following protocol of Jason, a seven-year-old child, we see a nonconventional divisional process.

- T: What number do you want to divide by now (Jason has just completed $24 \div 4$)?
J: Three
T: OK. I'll give you one with three (Presents the sentence $24 \div 3 = \underline{\quad}$). You try that one.
J: 3, 6, 9, 12 (sequentially puts up four fingers on his left hand. He then turns to his right hand and sequentially puts up three fingers because he doesn't know the next number word of the sequence) 13, 14, 15 (He then puts up the remaining finger of his left hand) 15. (Jason repeats putting up three fingers of his right hand to generate "18", "21", and "24", each time putting up a finger of his left hand and uttering the number word just generated. He then stops at "24" independently.) twenty-four -- eight!
T: Eight what?
J: Eight fours -- I mean, eight threes (emphatically)!

Jason's divisional scheme clearly involved counting by threes. When Jason counted by threes, "3, 6, 9, 12", each counting act was a symbol for a unit of three. This was illustrated by the manner in which he recorded counting by ones "13, 14, 15" to generate the next number word of the sequence.

We characterize Jason's divisional concept as being the application of his iterative scheme for counting by threes to the result of counting to 24 by ones. In quite simple terms, one meaning of the symbol "24" for Jason was an intention to act -- to count "1, 2, . . . , 24". Another meaning that he introduced was to count to 24 by threes which he could anticipate doing. We see that the latter meaning was based on counting by ones and, in fact, constituted a reorganization of it in that he could anticipate making units of three when counting.

An iterative multiplicative or divisional scheme is more than the activity of counting by threes or by fives, as the case may be. Coordination of two counting schemes, counting by ones and counting by an iterable unit, is also required. This anticipatory coordination is a mechanism the child uses to assimilate multiplicative or divisional situations and, as such, triggers counting by the particular iterable unit involved.

Limitations of Iterative Schemes.

Returning to Tyrone's "counting by twenties" solution to "30x20", an observer can "see" three quantities in Tyrone's activity -- the unit 20, the multiple 600, and the number which specifies what multiple 600 is of 20, namely 30. It might therefore be thought that the concept which triggered counting by twenties echoed Gauss' (1929) view that quantities are always defined by an indication of "how many times a known quantity (unit), or an aliquot part of it, must be repeated in order to obtain a quantity equal to the one that is to be defined, and that is to say, one expresses it by means of a number" (pp. 58-59). Gauss was thus able to specify a numerical relation further than merely as the relation between a whole and its parts. "We have to take into account three quantities, the unitary, the multiple, and the number that indicated what kind of a multiple it is" (pp. 59-60)².

Two characteristic breakdowns in Tyrone's problem solving behavior indicate that he did not view 600 as a multiple of 20. We have seen that Tyrone could relate successive tasks when the multiplier was varied. For example, he used the result of iterating 20 twenty times when solving "30x20". However, he invariably broke down when attempting to relate tasks when the multiplicand rather than the multiplier was varied even by one.

The second characteristic breakdown occurred when Tyrone attempted to solve division word problems. He could solve such problems by using his iterative scheme if the iterable unit was given and he was asked to find the number of times it could be repeated. However, he did not produce a response if the number of times an iterable unit was

to be repeated was given and he was asked to find the number of items in the unit. These breakdowns bring to mind Piaget's (1973) assertion that "the pupil will be far more capable of 'doing' and 'understanding in actions' than of expressing himself verbally. In other words, a large part of the structures the child uses when he sets out actively to solve a problem remain unconscious" (p. 86).

Abstract Multiplicative and Divisional Concepts

Our interest in iterative schemes should not be interpreted as implying that we view the child as a "counter" or that we intend to leave the child in such a state that his or her primary response to arithmetical problems is to count. Rather, our primary premise is that the iterative schemes are essential material in the construction of abstract concepts of multiplication and division, concepts that require complete abstraction from any particular context.³ This demanding reflective abstraction is similar to that which has to be achieved for addition and subtraction concepts before part - whole structures become available in spontaneous re-presentation (Brownell, 1928; Cobb, 1983; Labinowicz, 1982; Steffe, et al., 1983). Products like "7x8" will then refer to the operations involved in repeating a unit of eight abstract units seven times in order to produce a partitioned number. A particular implementation of this concept consists in giving multiplicative meaning to a seven by eight array of items. Another implementation is giving meaning to the activity of segmenting the number line for 1 to 56 in modules of eight. Similarly, quotients like "56÷8" will refer to the operations involved in constructing how many times a unit of eight abstract units must be repeated to partition the number 56. These concepts are not to be confused with what passes for repeated addition. Herscovics, Bergeron, and Kieran (1983) note that "repeated addition. . . reveals an emphasis on the formal definition of the arithmetical operation, a definition whose nature is far more procedural than conceptual" (p. 2).

1. This paper is based on work supported by the National Science Foundation under Grant No. SED 80-16562.

2. These quotations were translated from the original edition of Gauss' Werke by E. von Glasersfeld.
3. These concepts are conducive to the further elaboration of the multiplicative concept as consisting of a combinatorial composition.

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THE CONSTRUCTION OF ARITHMETICAL UNITS
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RESUME

L'observation de trois enfants de sept ans tentant de construire et d'utiliser la notion de dizaine, alors qu'ils se trouvaient au stade du dénombrement avec unités verbales, nous a amenés à distinguer plusieurs types de dizaines dans ce qu'ils ont bâti et utilisé. La dizaine est apparue comme unité perceptuelle, comme unité perceptuelle dénombrable, comme unité dénombrable au plan moteur, et comme "pattern" de mots désignant les nombres (ou unité corespondant à un acte moteur implicite). Aucun de ces types ne rejoint le concept abstrait de nombre.

Par contre, pour trois autres enfants de sept ans arrivés au stade du dénombrement avec unités abstraites, la dizaine apparaissait comme unité composée, comme unité abstraite, comme "dix de plus", comme unité itérative et comme unité de mesure. Ces cinq derniers types d'unité étaient tous "numériques" mais différents les uns des autres surtout en ce qui touche leurs implications.

THE CONSTRUCTION OF ARITHMETICAL UNITS¹

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Based on the documentary material of a two-year longitudinal teaching experiment, we have formulated various unit types that the six participating children constructed for the unit of ten. The formulation was aided by what, in previous work, has been established about the construction of the concept of unit in general (von Glasersfeld, 1981) and the concept of one as a counting unit and number (Steffe, von Glasersfeld, Richards, & Cobb, 1983). The formulation was aided also by an analysis of children's procedures when counting by different units (Thompson, 1982), by a philosophical analysis of the unit of ten (Richards & Carter, 1982), and by an analysis of the abstraction of counting units (von Glasersfeld, 1982). The apparent disparity in the conception and use of ten by the six children of the teaching experiment was the primary basis of the conceptual distinctions that are made in this paper.

The teaching method we used can be characterized as indirect guidance. We continually formulated and tested working hypotheses about each child's conceptions and procedures. This continual hypothesis

formulation and testing enabled us to build up a model of how each particular child constructed the concept of ten and the procedures used. The unit types are discussed below. The first four types were formulated to explain the construction and use of ten by Brenda, James, and Tarus when their counting by ones was based on verbal units. The last five unit types were formulated to explain the construction and use of ten by Jason, Scenetra, and Tyrone while they were in their abstract periods. All six children were seven years of age.

Ten in the Verbal Period

Ten as a perceptual unit. The first distinction that is made is between ten as a perceptual collection and ten as a perceptual unit. Perceptual collections are only temporarily established in the visual field of the child. Spatial location and spatial configuration are both critical features of the collections. For example, a child might judge a row of ten blocks as being more than a bag of ten blocks (even after counting each one) simply because the row is longer as Piaget, long ago, observed in his conservation experiments(cf. Piaget & Szeminska, 1941). The same child also might judge two such counted rows or two such counted bags as being "the same" because of their similar perceptual features. In fact, without counting them, the child might even say that a row of blocks or a bag of blocks has ten blocks if the perceptual features of the row or bag are like those of a previously counted row or bag. But changing location or configuration would lead to different judgments. Perceptual collections form a unitary whole, but from the perspective of the child, the perceptual items of the collections and the patterns in which they are arranged are salient. The child focuses on the contents of the collection rather than on the collection as one thing. It consists of an experientially bounded plurality which, as such, would be countable (von Glasersfeld, 1981).

If a child counts a perceptual collection prior to the ability to represent to itself counting events, the collection only temporarily embodies the counting acts. The activity of counting is something that the child introduces into the collection. Whenever a counted

collection triggers a re-presentation (von Glasersfeld, 1982) of counting -- playback of the counting activity -- the child can compare two collections on the basis of the re-presented counting sequence(s). This introduces "perceptual mobility" in the sense that the perceptual collection has been abstracted from its particular experiential context. While the counted perceptual items are still salient from the perspective of the child, any two perceptual collections of ten can now be viewed as having a common feature in that, if they were counted, "ten" would be the result. We say that the perceptual collections are "framed" by counting even though they are still viewed by the child as comprising ten individual items. Perceptual collections are akin to what Herscovics (1983) has identified as a unit in a physical sense: "The bag of candies is clearly discrete and experiential and remains a unit until the bag is broken" (p. 18). However, a perceptual unit of ten is not constrained by such physical boundaries.

Ten as a countable perceptual unit. Whenever a child coordinates the number word sequence "10, 20, 30, . . ." with specific perceptual units, we say that, for the child, these units are countable. Because it depends on an actual count of perceptual items, this conception of ten has specific limitations. They manifest themselves in the breakdowns that were observed. For example, Brenda could count visible bags of ten cubes each by tens, but whenever the bags were screened from her sight, she was unable to count them as tens. The perceptual collections had to be actually visible for her to count by tens.

James, who could create perceptual units as did Brenda, placed six cellophane bags of ten candies on six popsicle sticks. Touching each, he counted the bags of candy by tens, uttering "10, 20, 30, 40, 50, 60" in synchrony, and said "sixty". He then said there were six popsicle sticks. But when the teacher asked him if sixty was more than six, he said "No, the same amount". Here, "sixty" referred to the six bags, even though in the teaching episode "candies" were being referred to.

The teacher then picked up a single bag of candy and asked James how many candies were in the bag. James replied "ten". He then took the

popsicle sticks and said that he had six such sticks. He also replied that the teacher had more candies in the bag than he had popsicle sticks when asked who had more or if each had the same number. In this case, he focused on what the bag contained -- on the collection of candies. So, when James had more than one bag, he took the bags as singleton units corresponding to "one". But when he had a single bag of ten, he focused on its contents. What James could not do was to take the bags as singleton units while maintaining their numerosity. He could not "pack" a collection of counted perceptual items into a composite unit and flexibly move between that composite unit and its contents.

Ten as a countable motor unit. Ten as a countable motor unit occurred in the context of counting perceptual units of ten. A motor act like putting up a finger or pointing with a finger was used as a substitute for perceptual units of ten that were screened from sight. For example, when three bags of ten blocks were hidden by one cloth and five by another, James uttered "10, 20, 30, -- 40, 50, 60, 70, 80" in synchrony with putting up fingers. Counting was not extinguished. However, James insisted that there were eighty bags even though the teacher asked him whether there were eighty or eight. He also said that there were ten blocks in a bag, but he never said that there were eight bags and eighty blocks. Putting up a finger was a substitute for a bag taken as a perceptual unit without regard to its contents.

Putting up a finger was a substitute for the contents of a bag of ten items for Brenda. She could not keep track of counting the bags hidden by the second cloth. Focusing on the bags as collections and not as singleton units, she could not re-present them as elements of a spatial pattern (eg. a domino five) that could be used to keep track of counting acts. Both James and Brenda could coordinate putting up fingers (or closing fingers) with their number word sequence "10, 20, 30, . . . ". But they still could not focus on the unitariness of the perceptual units while maintaining their numerosity. This observation is consistent with their inability to make records of how many times they uttered number words in sequences like "4, 14, 24, 34, . . .".

Number word patterns. That it was appropriate to attribute re-presentation of counting activity to the three children was indicated by the way in which they abstracted number word patterns. Previous attempts at such abstractions when the children were in preceding periods -- perceptual and motor periods -- had been complete failures. The first occasion where Brenda abstracted number word patterns from counting by ones occurred in May of the second year of the teaching experiment. Her task was to find out how long a "bridge" was that consisted of, first, three centimeter cubes, and, second, eleven sticks of ten centimeter cubes, the last six of which she placed end-to-end while counting them. She counted the first three centimeter cubes "1, 2, 3". She then opened her hands and uttered "4, 5, 6, 7, 8, 9, 10, 11, 12, 13" in synchrony with putting down fingers. She repeated counting the sticks of ten centimeter cubes in this way until she had counted the first six. The teacher encouraged her to predict the number word she would say if she counted as she placed each stick. After her second placement, she sat and stared into space for about seven seconds and then blurted out "73!" After this, she went on uttering "83, 93, 103" in synchrony with her placements. On the last placement, she resorted to counting by ones because the number word pattern was broken. Until Brenda reached "63", there was no apparent abstraction or review of her past counting activity. Her attempt to predict the number word she would say last when counting, introduced a new element into her behavior. We believe that she re-presented and reviewed the results of the counting activity in her immediate past experience and pulled from it the recurrent results of making intuitive extensions of ten (which was the recurrent aspect of her experience). Our interpretation is based on her pause of about seven seconds while she stared into space and on what she subsequently did.

Brenda constructed a similar number word sequence in the very next task. In this task, one more centimeter block was placed with the row of three and she was asked to count again. This time she counted 1, 2, 3, 4; 5, 6, . . . , 14; 15, 16, . . . , 24; 25, 26, . . . , 34; 44, 54, 64, . . . , 104, 114, putting fingers down in each of the three intuitive extensions of ten. After the third extension, she

again abstracted the number word pattern and curtailed making the intuitive extension.

It is essential to consider what, say, "73" referred to when Brenda uttered it as the number word that would be the result of coordinating "64, 65, . . . , 72, 73" with closing fingers. We take her utterance of "73" as indication of her intention to count as described (cf. Van Engen, 1949). These unexpressed counting acts are what we call an implied motor unit of ten.

Ten in the Abstract Period

Ten as a composite unit. The ability to take a perceptual collection of ten items as one thing while maintaining its numerosity, constitutes a conceptual leap (Freudenthal, 1979, p. 56). We illustrate this leap by a protocol of Jason, another participant in our teaching experiment, solving a missing addend task. (In the protocol, "I" indicates the instructor and "strip" refers to eight inch strips of paper on which ten centimeter squares were glued.)

- I: Shut your eyes (places four strips under a cloth and three strips ~~by~~ the cloth). Now, open your eyes! There are seventy little squares altogether. How many strips are under here?
- J: Three (looks at the three visible strips), 4, 5, 6, 7 (looking at successive places on the cover that he takes as hiding strips) four.

Jason was able to take strips as units in spite of the fact that the number he had been given referred to centimeter squares as units. His ability to maintain the numerosity ten is demonstrated also in a task where he was shown three strips, was told that twenty squares were covered by a cloth, and was asked how many little squares there were altogether. He counted in modules of ten by uttering "31, 32, . . . , 40" in synchrony with putting up fingers and then repeated this process until he reached "fifty." Before he started counting, he organized the hidden squares into strips by two sweeping gestures over the cloth -- twenty squares make two strips. In other words, he re-presented two strips and then counted their contents.

This shows that he could take a strip as one thing while maintaining its numerosity, ten. It shows also, of course, that he could represent a perceptual unit of ten as a spatial pattern.

Ten as an abstract unit. The hallmark of ten as an abstract unit is the ability to coordinate counting by tens and ones when counting-on. For example, Tyrone counted by tens and ones to find how many dimes and how many pennies were covered when he was shown two dimes, told that there were 53 cents in all, and asked how many dimes and how many pennies were hidden by a cloth that screened three dimes and three pennies from his sight. He counted the two visible dimes "10, 20", and then continued by making a numerical extension. He uttered "30, 40, 50" in synchrony with putting up three fingers and then "51, 52, 53" in synchrony with putting up the same three fingers. He then knew that there were three dimes and three pennies covered.

Tyrone independently distinguished between an act of counting that involved a dime and an act of counting that involved a penny. A dime and a penny were not homogeneous countable items. Nevertheless, they were both taken as being countable items in the same counting activity. To do this, requires taking the composite unit, ten, and the unit, one, as objects of reflective thought. In other words, the two units of different ranks were material for a further uniting operation. A dime and a penny were equivalent from the perspective of their unity but were distinct from the perspective of their numerosity.

The difference between the abstract unit of ten and the composite unit of ten can be simply put. In the case of a composite unit, a child operates with discrete perceptual or figural items, either uniting them into a whole or decomposing the whole into its constituent parts. In the case of the abstract unit, the child becomes aware of the aggregate as a component and can distinguish it from other such components -- in particular, the component of one. But in neither case is the child able to survey a potential perceptual field and anticipate segmenting it into units of ten and counting those units.

Ten more. This unit type arises when the child, in contexts of counting-produces abstract units in order to constitute the numerosity "ten". Attention is directed on the repeated production of units of "one" rather than on the abstract units themselves. On one occasion, Tyrone segmented counting-on into modules of ten counting acts.

- I: (Places three strips in front of Tyrone and hides two strips and two more squares). How many are here (the three strips)?
- Ty: Thirty.
- I: If I put these (hidden squares) with them, there would be fifty-two. How many would be under here? How many little squares and how many strips?
- Ty: (Sequentially puts up twenty-two fingers in synchrony with subvocal utterances. He then says, after a lengthy pause where he appeared to be deep in thought) there would be two of these (slaps the strips) and two of them (gesturing toward a pile of squares).
- I: How did you figure that out so well?
- Ty: I counted, 10, 20, 30, (in synchrony with moving strips); and then 40, 50; and then 51, 52!

Counting by ones was partitioned into increments of ten as he went along. After Tyrone reached "52", he reviewed his past activity, capturing the distinctions that he had made. How he reviewed the activity is strongly indicated by his explanation "I counted 10, 20, 30; and then 40, 50; and then 51, 52". While this was not what he actually did, it was the meaning of what he did. Tyrone was obviously aware of counting by ones while counting. His capacity to hold that past activity at a distance and review it, pulling out its structure, is a strong indication of reflective abstraction.

Ten as an iterable unit. Reorganizing a numerical extension into modules of ten, when the extension is divorced from particular perceptual (or figurative) material, involves, prior to the actual counting activity, a decision to count by tens that is not suggested by say, a dime or some collection of

ten items. It is an internally generated decision based on the potential iterations of the unit. When iterating a unit of ten the counting acts form successive intervals, in that each counting act is a symbol for "ten more". Metaphorically, counting so many abstract units of ten is like jumping from one lattice of ten rungs to the next lattice, when the lattices are already in place. Iterating a unit of ten so many times is like laying down a lattice of ten rungs end-to-end, with the intention of finding out how many times it can be laid down.

When ten becomes an iterable unit, the child can survey a potential perceptual field and anticipate segmenting it into units of ten and counting those units. Prior to constructing ten as an iterable unit, the children could very well give meaning to a number as so many tens and so many ones (provided that the number was named by a two-digit numeral), but they could not spontaneously count by tens to find how many collections of ten they could make from, say, a pile of 100 centimeter cubes. They had to be led through the task laboriously, making rows of ten cubes per row. They could not even anticipate how many more rows of ten could be made after making, say, six such rows. All three of the children in the abstract period first developed the abstract unit of ten and only then did they construct the iterable unit. In a philosophical analysis of the iterable unit, Richards and Carter (1982) stated that for children to use "ten" as an iterable unit, "they must be able to strip it of its composite quality" (p. 61). That is one thing that has to be done and, in our analyses of the children's behavior, that is precisely what took place when ten became an abstract unit. Another thing that has to be done for ten to be used as an iterable unit is that it must be abstracted from an iteration of the unit of one.

Ten as a measurement unit. Counting can be a process of measurement, and ten and one can be units in a measurement system. The coordination of counting by tens and counting by ones when counting-on is a crucial step in the construction of such a system. Also crucial is the construction of ten as an iterable unit, for it provides the child with

foresight -- with an ability to anticipate the results of counting by tens.

Counting backward by tens and ones is required for ten to emerge as a measurement unit. In one case, Tyrone counted backward "61, 51, 41, 40, 39" to solve $71 - \underline{\quad} = 39$. He knew very well that putting up a finger in synchrony with uttering "61" symbolized ten backward counting acts and that, if he counted forward from "61" ten times, he would reach "71". This flexibility, introduced by counting backward by tens and ones, and reversibility of counting is crucial in the maturation of ten into a measurement unit. But it still is not sufficient. Tyrone, for example, was not yet able to count backward by ones from any point in the number sequence which had been reached by a count by tens. Only when this system of inclusions becomes operative for the child will ten as a measurement unit emerge, for "ten" as a counting unit would then be interchangeable with "one" as a counting unit.

Summary

The unit types can be summarized as follows:

1. Perceptual unit. Any perceptually bounded pattern of ten counted items that can be represented.
2. Countable perceptual unit. Any perceptual unit of ten that is coordinated with a number word of the sequence "10, 20, 30, ...".
3. Countable motor unit. Any motor act that is coordinated with a number word of the sequence "10, 20, 30, ..." and that constitutes a substitute for a perceptual unit.
4. Number word pattern. Any re-presented pattern of ten number words that can be uttered in sequence. In so far as the number words of the re-presented pattern imply curtailed motor acts, we also speak of an implied motor unit of ten.

5. Composite unit. Any pattern of ten items where the items retain their unitariness as well as their numerosity.
 6. Abstract unit. "Ten" as a conceptual structure that permits the coordination of a count by tens and a count by ones when making a numerical extension (i.e. in counting-on so many tens and so many ones from a given number).
 7. Ten more. "Ten" as a module of ten counting acts that can be repeated when making a numerical extension.
 8. Iterable unit. "Ten" as a conceptual structure that is available in the absence of particular collections of ten perceptual items and makes possible the decision to count by tens prior to the actual activity of counting perceptual or hidden objects.
 9. Unit of Measurement. "Ten" as a counting unit that is freely interchangeable with "one" as a counting unit, and is available as counting unit irrespective of the point in the number word sequence where the count begins or the direction of counting (forward or backward). Reversible coordinations of the two units are present.
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FROM ACOUSTIC COUNTING TO NUMEROSITY

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RESUME

En observant de petits enfants (4 à 7 ans) comptant des ensembles, on remarque des erreurs qui semblent témoigner d'idées sur le sens du comptage qui diffèrent des nôtres. Par exemple:

Les petits ne montrent pas de surprise si un second comptage du même ensemble conduit à un autre résultat.

Quelques-uns comptent toujours de nouveau pour répondre à la question "combien"?

Indiquer les objets et réciter le rythme des nombres, ce sont des processus mal synchronisés.

D'autre part le comptage est strictement borné à la récitation des 1,2,3, ...

L'élément commun de ces phénomènes est que, quand ils comptent, le mouvement et le son sont primaires tandis que le nombre des objets est secondaire. D'après notre recherche le comptage acoustique peut pousser le mouvement en avant, indépendamment du nombre des objets visuels à compter.

Pour mettre l'accent sur le nombre on doit chercher des mécanismes appropriés pour maîtriser la force poussante du comptage en l'accélérant, en le ralentissant, ou en changeant son rythme.

Pour conduire l'enfant du nombre comptant au nombre cardinal, il est utile de se servir d'artifices de freinage arrêtant le flux du comptage. Voilà le sujet de notre recherche.

PROPOSITION

When counting quantities of objects young children (4-7 year-olds) make mistakes which lead us to believe that these children have deviating ideas about counting compared to ours.

Examples:

. Infants are not surprised that a second count of given objects produce a different count to the first.

They are evidently focused on counting as pure movement,

regardless of quantity.

- . When asked repeatedly 'How many is that?' these children demonstrate the same misconception by counting the objects all over again.
- . Sometimes objects are omitted and therefore not counted. This is 'wrong'. But in counting games, games entailing counting, omission is the rule rather than the exception.
- . The indication of objects and the accompanying rhythmical recital of numerals does not usually occur synchronically.
In children there seems to be no essential link between these two activities.
- . Sporadically it occurs that a child is unable to remember the sequence of numbers.
Improvement can be brought about by the rhythmical use of percussion instruments.
- . Often the children misuse the acoustical sequence of numbers.
Instead of counting per whole numeral, they count per syllable.
E.g. se-ven means 2 objects, e-lev-en means 3 objects.
- . The great difficulty is that children cannot dissociate themselves from counting in the form of 1, 2, 3, --- This impeded them from seeing numbers structurally.

Taking all these factors into consideration it would seem that in the first place counting has to do with *movement* and *sound* and only in the second place with *objects*.

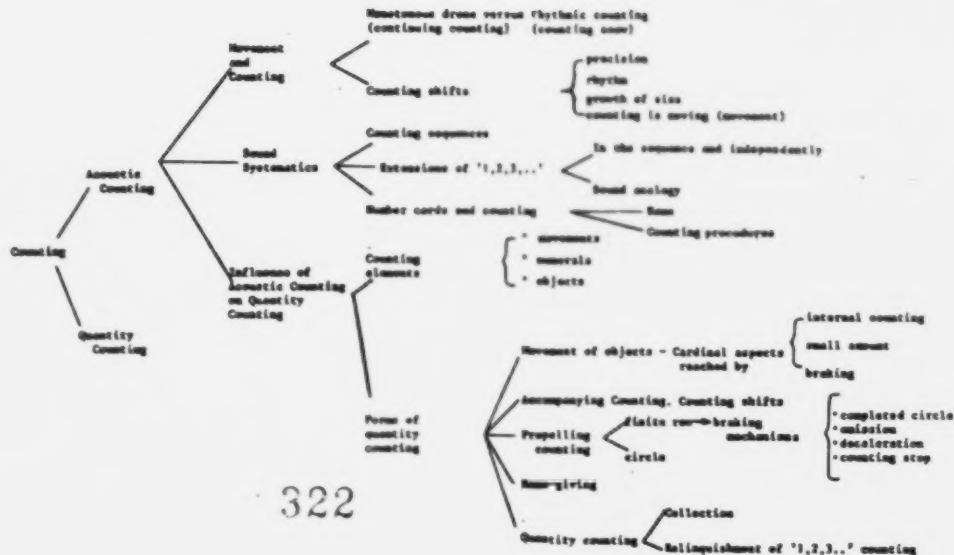
In this article we advance the opinion that *acoustic counting*, i.e. mechanical rhythmical counting *without* objects, is developed previous to counting of quantities of objects. In this light, the misunderstandings which these children have about counting quantities becomes comprehensible.

Within the conception of acoustic counting there are two separate areas of research:

- . The mutual influence of movement and counting
- . The sound systematics of numerals

From this investigation we would like to give some recommendations for the benefit of schools on the subject of counting. These are

particularly relevant to the control of counting as movement (slowing down, quickening, stopping.).
Perhaps gilding the lily, we give you a diagram of the above ideas:



COUNTING

Counting is not limited to the counting of quantities.

It is used by children along with movement, games and singing and just to pass the time.

This kind of counting is first and foremost of an acoustic nature. The counting of quantities is an application thereof.

Acoustic counting is the recital of a sequence of numbers in a certain order.

This is a broad definition as the sequence does not necessarily need to be the natural or 'expected' sequence nor does it need to begin with 'one'.

Neither are quantities of objects necessary in order to give the answer to the question: 'Can you count?'

For that matter, not every researcher will define counting in these terms.

Some researchers will understand counting exclusively as the counting of quantities. (Steffe, c.s., 1980, p. 102; Richards, 1981, p. 25; Mierkiewicz and Siegler, 1980, p. 250).

But there are also researchers who share the opinion that acoustic recitative counting precedes the more specific counting of quantities. (Gerlach, 1914; Kühnel, 1916, p. 37; Fuson, 1980, p. 257; Steffe, c.a. 1980, p. 102, 103)

Let us now look into a few differences between acoustic counting and quantity counting.

i.e.

- . Sounds may not be omitted in quantity counting.

After '10' comes '11' and not '20'

- . No numerals may be said twice

- . You must begin with sound 'one'

- . You must go on counting to the end of the row of visible objects

Acoustic counting differs in these phenomenons.

ACOUSTIC COUNTING

The phenomena that arose in our investigation of acoustic counting in 4 to 8 year-old children can be divided under three headings:

- 1) The *dynamics* of acoustic counting or the interrelation between counting and movement
- 2) The sound systematics of acoustic counting
- 3) The influence of acoustic counting on *quantity counting*.

1) *The dynamics of acoustic counting*

We shall discuss two aspects:

- a) monotonous counting as opposed to rhythmic counting
- b) count shifting

a) *monotonous counting versus rhythmic counting*

Some investigators consider that rhythm and measure are derived from physiological processes in man and from movement in space.

(Lievegoed, 1939, p. 57 e.v.; Bindel, 1982, p. 70)

On the other hand, we are shown that the lexical meaning of words often escapes young children and that the external appearance of the words, i.e. rhythm and measure, is what gives them significance. Rhythm and measure have different functions in regard to acoustic counting.

b. *Count Shifting*

Another phenomenon is that counting and movement do not happen simultaneously.

In the past different causes for this phenomenon have been indicated, such as:

- . lack of accuracy (Ginsburg, 1977 a, p. 7)
- . lack of feeling for rhythm (Baldwin and Strecher, 1925, p. 141)
- . lack of insight into the notion that 'counting one further' means 'counting one more'

(Kühnel, 1916, p. 33), by which the link between pointing and counting is absent.

We think that this is caused by the fact that children's counting approximates more to acoustic counting (i.e. counting which is largely dependant on speeds of movement and enunciation) than quantity counting.

2) *The sound systematics of acoustic counting.*

In this section we shall go further into the meaning of different sequences of numbers (1,2,3,...; 10,20,30,...; 5, 15, 25,...; 100,200,300,...).

We shall look at the attempts of children to extend the number sequence 1,2,3,... and shall discuss the connection between counting and written numbers.

3) *The influences of acoustic counting on quantity counting*

In order to count quantities of objects, a child must combine three elements of counting:

- a. The movements necessary for pointing out or moving objects
- b. The numerals that must be expressed
- c. The objects to be counted

(Mierkiewicz and Siegler, 1980, p. 249-256)

In acoustic counting elements a and b are central:

- a) dynamics and b) sound systematics

In the counting of quantities of objects is to be regarded as the material appearance of acoustic counting.

Acoustic counting is made more specific:

You must begin with the sound '1' etc.

In the nature of the case the emphasis is now more on visuo-graphic counting, on the 'seeing' of collections and structures, than on movements and sounds.

On the other hand, movements and sounds (rhythms and measure, count shifting, knowledge and use of sequences of numerals, etc.) exercise their influence on this specific form of counting. (quantity counting).

In fact the counting child emphasizes one of these three elements a, b, or c, or uses a combination thereof.

And this determines what a certain child at a certain moment understand as 'the quantity counting'.

Theoretically, one can distinguish 7 different combinations of the 3 counting elements (a, b, c) and therefore 7 different forms of quantity counting. Some of which are nearer acoustic counting,

(elements a or b of a combination of these two) than to quantity counting, (a combination of a, b and c).

In our investigation we have subjected five forms of quantity counting to a further scrutiny.

I *Moving* (counting element a is central)

'Quantity counting' in 4-ages olds, for example, is nothing more than the task of moving:

moving the eye or the finger, moving quantities of objects.

This is accentuated in the counting of non-stationary collections of objects, as when children are counted in the playground

II *Accompanying counting*. (counting elements a and b)

In contrast to III, movement (a) can remain primary and counting is regarded as the accompaniment to the pointing (i.e. moving of the finger)

III *Propelling counting* (counting element b is central).

Here, the pointing movements are counted.

The counting propels, as it were, the movements along.

IV *Labelling*. (counting elements a and c)

Counting quantities is giving a name to each object; the name being a numeral which is given to each object separately, not to the whole collection.

V *Proper object counting* (a, b and c)

It is not the counting or the movements which determine and limit the counting itself, but the collection of objects - the structure, visible or invisible parts of the collection, etc.

These five forms of counting quantities came to light by bringing children into conflict situations in which a certain aspect of quantity counting became unmanageable.

We shall look at the form *Propelling counting*.

The mastery of the acoustic sequence of numerals in word and rhythm proceeds at an increasingly rapid pace (Fuson, 1980, p. 257)

This causes acoustic counting to produce a propelling effect

on the movements (pointing, moving of objects) which are performed in the counting of objects. Counting faults are the direct result of this propelling. The end of the row of objects for instance is apparently a visual phenomenon which shrinks into insignificance compared to the continuing acoustic propelling counting.

B&T, the end of the row just determines the quantity! According to our research, we can conclude that the braking or acoustic counting, before the end of the row is reached, is obviously the way of making the cardinal number break through to consciousness. In order to investigate the mechanisms of braking I presented the infants various situations which focus their attention on number:

- . if you go on counting round a circle (e.g. string of beads) the children will ultimately come to a halt;
- . omission of objects is not acceptable and therefore acts as a brake;
- . the acceleration or deceleration of acoustic counting and the accompanying synchronous pointing;
- . The cessation of counting before all the objects have been counted, leads to the retention of the reached-number(counting stops are important).

Taking these mechanisms into consideration it can be seen that braking plays a large part in converting acoustic propelling counting into a method for determining quantities.

Note

A paper will be available at the conference.

F. GEOMETRY AND SPATIAL REPRESENTATION /

GEOMETRIE ET REPRESENTATION SPATIALE

VERS UNE DÉFINITION OPÉRATOIRE DE LA PERCEPTION SPATIALE⁽¹⁾

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Dominique Dion, étudiant, U. de M.
Richard Pallascio, Collège Edouard-Montpetit
Vincent Papillon, Collège Jean-de-Brébeuf

SUMMARY

With Lohman who conclude, after a comprehensive review of the spatial ability literature, that "spatial ability may be defined as the ability to generate, retain, and manipulate abstract spatial images"⁽²⁾, we conceive the spatial perception from a structural or operative point of view involving the qualitative interiorisation of a spatial model by the analysis and the synthesis of his properties which are, orderly, topologic, projective, affine and metric. It is the reason why our concerned abilities' typology in this spatial perception, proceed from comprehension to intervention, analysis to synthesis, imaginary to imagination, intellectual plan to creative plan, with the following correspondent operations and abilities:

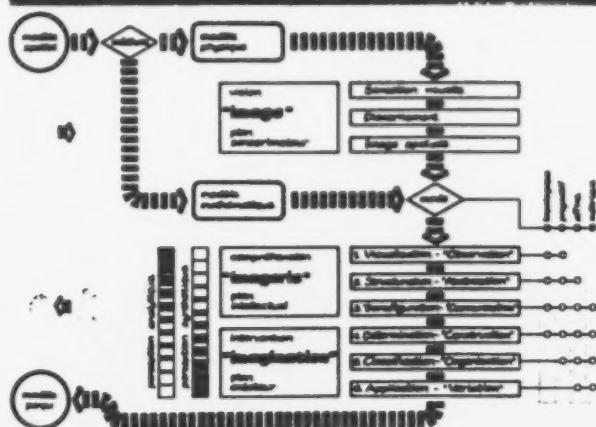
OPERATION	ABILITY
visualisation	observation
structuration	abstraction
transfiguration	communication
détermination	construction
classification	organisation
application	variation

In the following pages, we briefly illustrate this operative definition of the spatial perception and we present a serie of questions, non again validate, on the left quadrilateral. We finish this written presentation by some results.

(1) Première partie d'une recherche subventionnée par le ministère de l'Éducation du Québec, F.C.A.C. 84-AR-0006, intitulée "Identification des facteurs composant l'habileté à percevoir l'espace et de moyens permettant de la développer, auprès de jeunes de 9 à 18 ans."

(2) Reported by Ken Clements, see Reference [3], p. 35.

Perception Structurale / organigramme



- 330

EXCURSION DANS L'ESPACE

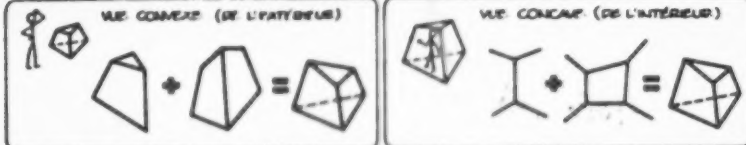
Les illustrations qui suivent vont nous servir à présenter notre typologie concernant l'habileté à percevoir l'espace, pour l'instant hypothétique. Le traitement de l'information comprend six étapes mentionnées dans l'organigramme précédent impliquant pour chacune d'elles, une action ou opération définie à l'aide de sous-étapes, de même que l'habileté qui y correspond. Notre excursion dans l'espace va consister à illustrer la perception structurale d'un prisme triangulaire, représenté à l'aide de l'un ou l'autre des média suivants: physique, linguistique, géométrique ou algébrique, et dans un ou l'autre des modes suivants: topologique, projectif, affine ou métrique.

Représentations d'un modèle spatial

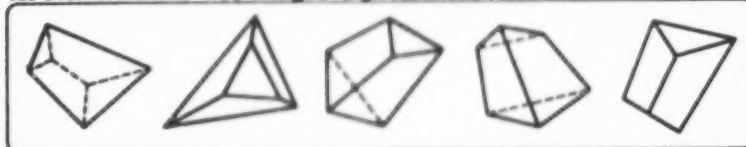
		MODES			
		topologique	projectif	affine	métrique
MÉDIA	physique				
	linguistique	Les liens entre six sommets A, B, C, D, E, F. AB, BC, AC, DE, EF, DF, AD, DE et FC.	Tétrahédre triangul.	Le prisme triangulaire.	Le prisme triangulaire semi-régulier.
	géométrique				
	algébrique				

1. Visualisation (Observation)

1a. Intégration des images partielles



1b. Mécanisation des images dérivantes

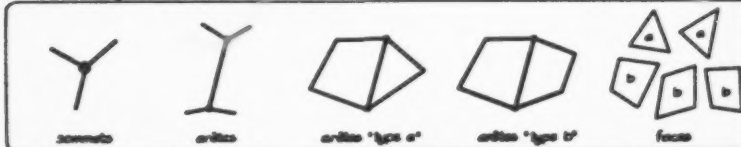


1c. Formation des images composées

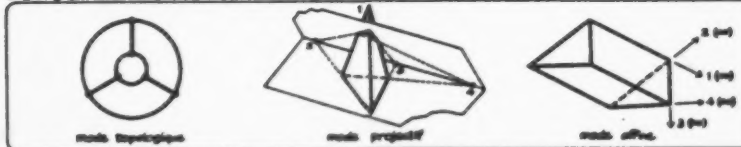


2. Structuration (Abstraction)

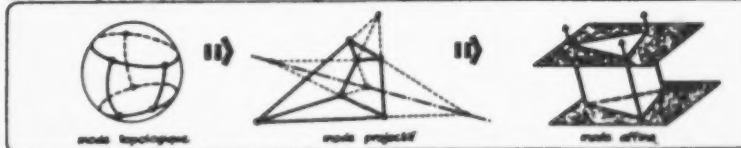
2a. Description des incidences



2b. Combinatorisation des incidences



2c. Construction des images géométriques



3. Transfiguration (Communication)

3a. Pécision sur la représentation

MODÈLE DONNÉ :

MODÈLE DÉSER :

3b. Codification des éléments

SOMMET

ARÊTE

FACE

3c. Représentation du modèle

4. Détermination (Construction)

4a. Énumération des invariants

Exemple : invariants métriques - Les 146 paramètres

DISTANCES

ANSES

4b. Figureation de la caractéristique déterminante (C_d)

no	A	T	2	lignes de choix
1	1	1	1	3
2	1	1	1	3
3	1	1	1	3
4	1	1	1	3
5	1	1	1	3
6	1	1	1	3
7	1	1	1	3
8	1	1	1	3
9	1	1	1	3
10	1	1	1	3
11	1	1	1	3
12	1	1	1	3
13	1	1	1	3
14	1	1	1	3
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38	1	1	1	3
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139	1	1	1	3
140	1	1	1	3
141	1	1	1	3
142	1	1	1	3
143	1	1	1	3
144	1	1	1	3
145	1	1	1	3
146	1	1	1	3

$C_d = 15 - 6^2 = 9$

$C_{146} = 7,32 \cdot 10^9$

9 Nombres de choix de coordonnées pour la position globale.

4c. Sélection des invariants indépendants

distances :

12 23 13

14 46 36

angles :

14 13

123 134

14 43

4, 23, 13, 14, 46, 36

14 13

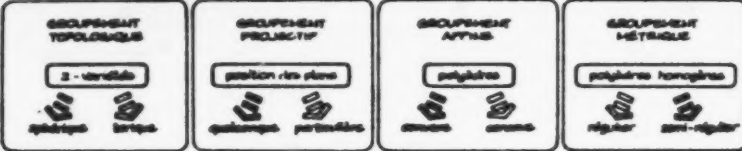
(deux additionnels)

123 134

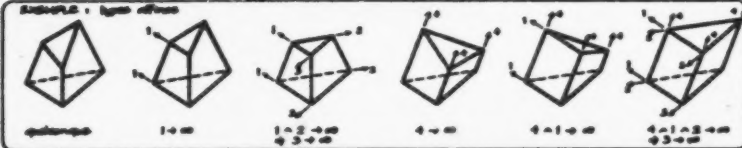
14 43

5. Classification (Organisation)

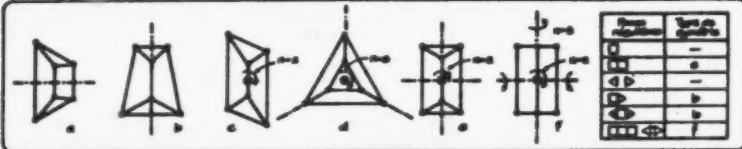
5a. Inclusion multiple



5b. Périovation des types combinatoires

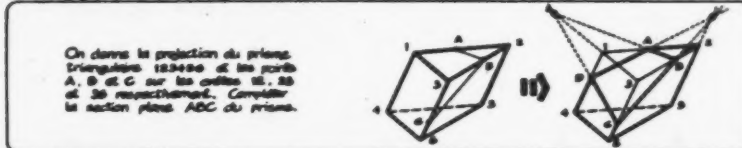


5c. Symétrisation - Régularisation (Congruence)

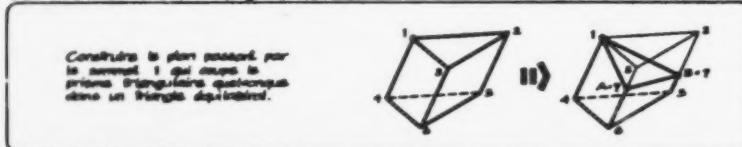


6. Application (variation)

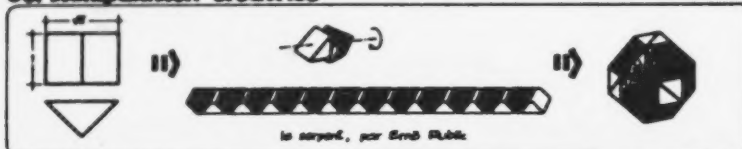
6a. Répétition des problèmes



6b. Formulation des problèmes



6c. Manipulation créatrice

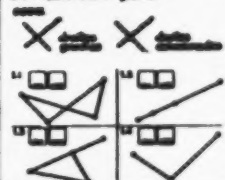


PREMIERS RÉSULTATS

Questions / Quadrilatère gauche

1. Visualisation

Déclarer si les diagrammes peuvent être ou ne peuvent pas être les vues d'un quadrilatère gauche.



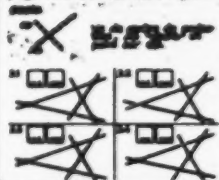
2. Structuration

Quatre points ont dans un même plan différentes...

- 11 ☐ parallèle
- 12 ☐ droite (d)
- 13 ☐ quadrilatère (et gonion(s))
- 14 ☐ tétraèdre(s)

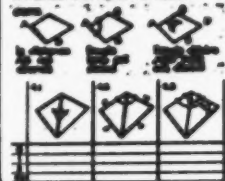
3. Transfiguration

Déclarer si les diagrammes peuvent être ou ne peuvent pas être les projections d'un quadrilatère gauche.



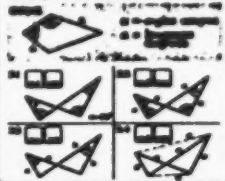
4. Détermination

Construire des quadrilatères gauches différents, mais différents par les propriétés métriques données ?



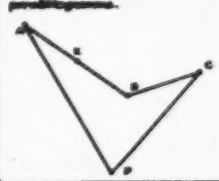
5. Classification

Établir si il des classes de quadrilatères gauches après les propriétés métriques données ?



6. Application

Construire le schéma plus précis par le point K, de coupe le quadrilatère gauche dans un parallélogramme.



Résultats en pourcentage (%) au test portant sur le quadrilatère gauche, passé à l'hiver 1983 auprès d'étudiants de premier cycle (âge moyen: 19 ans) inscrits à un cours de géométrie descriptive, à l'Université de Montréal (54 garçons et 35 filles).

	Garçons	Filles	Total (n = 89)
Visualisation	81	74	78
Structuration	20	8	15
Transfiguration	75	63	68
Détermination	14	16	15
Classification	48	46	46
Application	6	0	3
TOTAL	46	40	44

Les premiers résultats cités ci-dessus proviennent d'un test non standardisé et par conséquent non validé. Nous pouvons toutefois remarquer qu'il semble confirmer une habileté à percevoir l'espace plus développée chez les garçons que chez les filles, facteur déjà établi par d'autres chercheurs (Wood, 1977; Eliot et Fralley, 1976; Fennema et Sherman, 1977; Ernest, 1976). De plus les questions 1,

3 et 5, construites sur un mode binaire, laissent voir une diminution exponentielle dans les performances rencontrées (78, 68 et 46%), autant chez les garçons (81, 75 et 48%) que chez les filles (74, 63 et 46%). La structure temporaire des autres questions ne permet pas d'inférer sur celles-ci.

D'autres tests pré-expérimentaux portant sur la reconnaissance morphologique de différents polyèdres, tous issus d'un tétraèdre tronqué selon trois différentes coupes engendrant trois familles de polyèdres topologiquement distinctes, permettent de constater qu'il y a peu de différences dans les performances chez les garçons et chez les filles lorsqu'ils sont au primaire, et qu'il y a peu de différences chez les enfants et chez les jeunes adultes, sauf lorsqu'une performance linguistique facilite la résolution d'un problème.

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- [2] Baracs, Janos et Pallascio, Richard, "Le développement de la perception spatiale", in *Bulletin AMQ*, 1982, vol. 21, no 4, p. 5-11.
- [3] Clements, Ken, "Visual Imagery and School Mathematics", in *For the Learning of Mathematics*, vol. 2, no 2, nov. 1981, p. 2-9, et no 3, mars 1982, p. 33-39.
- [4] Dion, Dominique, *La perception spatiale*, Atelier de recherche AME 6410, Ecole d'Architecture, Université de Montréal, mai 1982, 4 p.
- [5] Pallascio, Richard, "La perception spatiale géométrique", in *Processus de géométrisation et de visualisation*, compte rendu de la 33^e rencontre de la CIEAEM, Italie, août 1981, pp. 37-50.
- [6] Piaget, Jean et Inhelder, Bärbel, *La représentation de l'espace chez l'enfant*, Paris, P.U.F., 1948.

REPRESENTATIONS ON PAPER OF THREE-DIMENSIONAL GEOMETRICAL SHAPES

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Résumé

Durant leur présentation, les auteurs feront part de certaines conclusions obtenues à la suite de deux études pilotes qu'ils ont récemment réalisées avec une importante population d'élèves de 4^e, 6^e, 7^e, 9^e et 11^e années de scolarité.

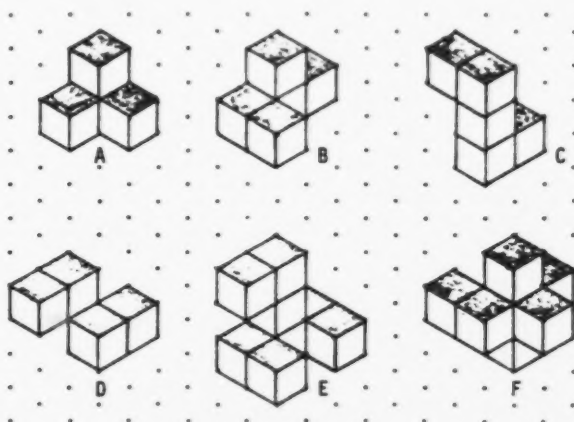
Le principal objectif visé était d'étudier les types de productions (verbales, graphiques ou mixtes) fournies par des élèves confrontés avec le problème de représenter sur une feuille de papier — au moyen de mots et/ou de dessins — une forme géométrique simple formée de cubes congrus juxtaposés.

A cette fin, une épreuve écrite a été administrée dans 21 classes, puis des entrevues individuelles ont été réalisées avec quatre ou cinq élèves par classe. Lors de l'épreuve, chaque sujet a reçu une forme géométrique (parmi six possibles) et une consigne (parmi deux possibles); il devait alors préparer un message sur une feuille blanche pour expliquer à un ami fictif comment se fabriquer une telle forme (ou encore, tout dépendant de la consigne, comment la reconnaître).

[Recherches subventionnées par l'Université Laval (P.P.M.M.) et par l'Université de Montréal (CAFIR) respectivement.]

This presentation is about two pilot studies that the authors have conducted concurrently in Quebec City and in Montreal⁽¹⁾ during 1982-1983. Approximately 500 pupils were involved, representing 21 classes from five public schools and one private school, and about equally distributed between 4th grade (10/11-year olders), 6th grade (12/13-year olders), 7th grade (13/14-year olders), 9th grade (15/16-year olders) and 11th grade (17/18-year olders).

The following task was administered individually to whole classes of pupils at a time. Each subject was given one three-dimensional geometrical solid made up of plastic cubes of edge 2 cm glued together. The six solids illustrated below were about equally distributed among the pupils.



Together with one plastic shape, each subject received one sheet of plain paper containing a short instruction and much space to give an answer. Half of the pupils received the following instruction

(1) The studies have been made possible thanks to the financial support of Université Laval (P.P.M.M.) and Université de Montréal (CAFIR) respectively.

(of an "algorithmic" type)⁽²⁾:

Suppose that one of your friends lives in France. He has a whole box of plastic tubes, all of the same size. You would like your friend to build a shape like the one you have in front of you. Prepare a message that you could send to your friend so that he may be able to build it. (You may choose to give explanations with words or drawings, as you like.)

while the other half received the following one (of a "descriptive" type)⁽²⁾:

Suppose that one of your friends lives in Ottawa. Near his home, there is a store selling all kinds of shapes made with plastic cubes. You would like your friend to buy you one shape like the one you have in front of you. Prepare a message that you could send to your friend so that he may be able to recognize that shape. (You may choose to give explanations with words or drawings, as you like.)

The pupils were told to read the instruction twice before answering and to work individually and silently. They were told also that it would not count for marks and that they would have much time to reply. No more explanations were given and no questions were allowed. On the average, it took about 35 minutes to administer the task with a whole class.

Essentially, the task of "preparing a message" amounted to representing a three-dimensional geometrical shape on a sheet of paper, i.e. on a two-dimensional surface. It was in order to make it more stimulating that it was presented as a communication task⁽³⁾. To make sure that the subjects would give a sufficiently broad meaning to the term "message", it was specified in the instruction that "You may choose to give explanations with words or drawings, as you like".

(2) The two instructions have been translated from French here. Notice that the word "message" may well be interpreted in different ways by Francophone and Anglophone pupils.

(3) During the progress of the study, the authors became aware that a similar communication task had been used in France in 1981 as part of a series of experiments. Cf. A. Bessot & M. Eberhard, *Représentation d'assemblages de cubes au C.M.*, published by IREM de Grenoble in the March 1982 issue of Grand N.

The main purposes of the two pilot studies were the following:

1. To study and classify the productions made by pupils of various grade levels when confronted with the task of representing a simple three-dimensional geometrical shape on a sheet of paper using words and/or drawings.
2. To study to what extent each type of representation identified above depends upon or varies according to
 - the type of instruction used ("algorithmical" or "descriptive");
 - the geometrical shape used (A, B, C, D, E or F);
 - the grade level or age of the pupils involved (grades 4, 6, 7, 9 or 11).
3. To study the various types of drawings produced by the subjects (as answers or parts of answers), the kinds of graphical "errors" observed and the "stages of graphical representation" that might eventually appear to emerge (analogous to those described by Lewis⁽⁴⁾ or Mitchelmore⁽⁵⁾).

In addition to the administration of the task in 21 classes, individual interviews of 10 to 15 minutes were conducted with four or five pupils from each class, in order to allow a deeper analysis of the results.

During the presentation, the authors will report about some of the conclusions of these pilot studies and they will give some suggestions for further research.

(4) Cf. H.P. Lewis, Developmental Stages in Children's Representation of Spatial Relations in Drawings, Studies in Art Education, vol. 3 (2), 1962.

(5) Cf. M.C. Mitchelmore, Developmental Stages in Children's Representation of Regular Solid Figures, Journal of Genetic Psychology, 133, 1978.

USE OF NEGATIVE INSTANCES IN IDENTIFYING GEOMETRIC FEATURES

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RÉSUMÉ

Il existe différentes théories sur l'utilité de cas négatifs ou de non-exemples. Dans une tâche où celui qui apprend doit identifier les caractères pertinents d'un concept, la recherche en psychologie privilégie l'emploi de tous les cas positifs. La recherche en éducation, pour sa part, supporte la théorie selon laquelle la combinaison de cas positifs et négatifs est plus féconde. Une recherche récente utilisant une tâche de chaîne de lettres (letter-string) a montré que la fréquence d'un caractère peut constituer une variable importante pour expliquer les positions différentes adoptées par la recherche en éducation, d'une part, et par celle en psychologie, d'autre part. Lorsqu'apparaît fréquemment (dans 90% des cas) un caractère qui n'est pas en rapport avec un concept, les étudiants peuvent être amenés à penser que ce caractère est nécessaire au concept. La présentation de cas négatifs peut contribuer à corriger ce malentendu.

Dans le but d'étudier la variable "fréquence du caractère" dans un contexte moins abstrait et plus mathématique, une tâche géométrique parallèle à la tâche de chaîne de lettres a été créée. Les résultats obtenus à partir de la tâche géométrique viennent appuyer les conclusions de recherches antérieures indiquant qu'il existe une interaction significative entre la fréquence du caractère (90% ou 50%) et le type de cas (positifs ou minas) utilisés.

La tâche géométrique a aussi permis de découvrir des problèmes intéressants qui surviennent lorsqu'un contenu académique, plutôt qu'une tâche abstraite de chaîne de lettres, est utilisé. Suite aux données et aux entrevues faites avec les étudiants, il est apparu que les cas négatifs peuvent être utiles à l'évaluation de la connaissance qu'ont les étudiants des caractères individuels d'un concept. Nous procédons à une révision du schème expérimental dans le but de vérifier cette hypothèse.

The role of negative instances (nonexamples) in concept learning has been of interest to the research community for many years (Bourne and Dominowski, 1972; Shumway, 1974, 1977; Sowder, 1980). The research has generated a variety of theories as well as several definitions of concept. The following definition was influenced by the work of Bourne and Dominowski (1972), Markle and Tieman (1970), and Shumway (1974).

A concept is a triple consisting of:

- 1) a universe of instances
- 2) a set of relevant features
- 3) a set of rules defining the relationship between the relevant features

In the concept "red triangular shape", the universe of instances could be "polygon shapes." The relevant features are "red" and "triangular." The relationship between the relevant features is "conjunction" (and). Some concepts may be more complicated and contain many relevant features and corresponding relationships but the given definition will generalize to any class concept. Concept learning can be evaluated by asking a subject to divide the universe of instances into positive and negative instances. For example, the concept of "red triangular shape" has been learned if the subject can separate given polygon shapes into red triangular shapes (positive instances) and shapes that are not both red and triangular (negative instances).

Researchers agree that a concept can be learned by seeing many positive instances, but there has been some disagreement on the use of negative instances. Psychological research favors the use of all positive instances in a task where the learner must identify the relevant features in a conjunctive concept. (Bourne and Dominowski, 1972; Erickson & Jones, 1978). In contrast, educational researchers found that sequences of mixed positive and negative instances were more helpful than sequences of all positive instances (mathematics, Shumway, 1974, 1977; poetry, Tennyson, Woolley and Merrill, 1972; chemistry, Tennyson, Steve and Boutwell, 1975).

Feature frequency was found to be a variable that could be responsible for the apparent difference between the psychological and educational research (Shumway et al., 1983). When an irrelevant feature of a concept appears frequently (90% of instances), students may think the irrelevant feature is necessary to the concept. Negative instances can be helpful in destroying the misconception.

The concept altitude of a triangle was structured to parallel the letter-string task used by Shumway and others (1983) and still maintain the characteristics of a common classroom concept.

The concept of altitude of a triangle has the following:

- 1) universe: line segments presented with triangles
- 2) relevant features: a) originating at the vertex
b) terminating perpendicular to a line containing the side opposite that vertex.
- 3) rule of relationship: conjunction

The irrelevant features were generated from common misconceptions about altitude that occur in the classroom. They include 1) orientation of triangle; 2) slope of the proposed altitude (vertical or not vertical); 3) location of proposed altitude (interior or exterior). The variables of feature frequency and presence of negative instances were investigated.

METHOD

Design. A feature identification task was given to algebra students in the form of instruments containing sequences of stimuli. The stimuli were figures containing triangles and proposed altitudes. Each figure was followed by the appropriate feedback to indicate that the line segment was an altitude (+) or that it was not an altitude (-). Subjects were asked to identify the two relevant features of the concept. The dependent variable was the number of trials before the correct pair of relevant features was reported consistently.

Four treatment groups were formed by crossing the two independent variables of feature frequency and type of sequence. The two frequency conditions were:

- 1) irrelevant features are equally likely to appear (1:1)
- 2) irrelevant features are weighted so some features appear 90% of the time and some features appear only 10% of the times (9:1)

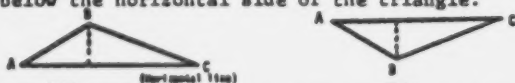
The two types of sequences were:

- 1) sequences of all positive instances
- 2) sequences of mixed positive and negative instances

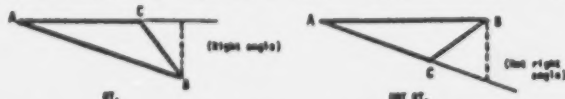
Each of the resulting four cells contained five sequences randomly generated within the restrictions of the treatment.

Procedure. There were two sessions with the subjects. The first session was designed to teach subjects how to use information from negative instances and how to use the evaluative instrument. The first session was taught using letter-string concepts and ended with a pretest. The second session was held the following day and involved the geometric concept altitude of a triangle. Subjects were not given the name of the concept but they were told it was a binary conjunctive concept. Five pairs of features were discussed with subjects and they were told to identify the two relevant features.

1. ABOVE or BELOW. The triangle containing the line segment was above or below the horizontal side of the triangle.



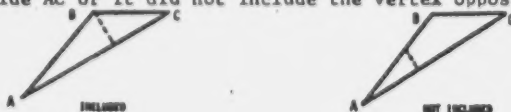
2. RT or NOT RT. The angle formed by the dashed line segment and the line containing side AC was a right angle or it was not a right angle.



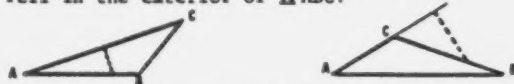
3. VERTICAL or SLANTED. The dashed line was vertical on the page or it was slanted.



4. INCLUDED or NOT INCLUDED. The dashed line included the vertex opposite side AC or it did not include the vertex opposite AC.



5. INTERIOR or EXTERIOR. The dashed line fell in the interior of $\triangle ABC$ or it fell in the exterior of $\triangle ABC$.



Subjects recorded the features present in each of 40 instances on a record sheet. After viewing feedback as to whether the instances were positive or negative students recorded their "guess" of the two relevant features.

RESULTS

The pretest indicated that most students understood the use of negative instances and that there was no significant differences between treatment groups. A sample of 100 algebra students who had reached criterion was used for the statistical analysis. The students were drawn randomly with the requirement that each sequence be represented by an equal number of students.

The mean trials to criterion and standard deviations for the students performance on each of the treatment groups is reported in Table 1.

Table 1. Means and standard deviations of trials to criterion for successful students on triangle task.

		Sequence Condition			
		+		+	
Frequency Condition		x	sd	x	sd
	1:1	4.8 n=25	2.7	9.4 n=25	6.5
	9:1	14.1 n=25	13.1	11.3 n=25	9.8

The 2x2 analysis of variance resulted in a significant F-ratio for the interaction between the sequence condition (+,+) and the feature frequency condition (1:1, 9:1) ($F(1.96) = 4.34, p < .04$). The

interaction confirms and potentially explains the different results of the psychological research favoring sequences of all positive instances and the educational research favoring sequences of mixed positive and negative instances.

DISCUSSION

In most classroom concepts, there are several irrelevant features that appear frequently. In textbooks, polygons are usually oriented with one side parallel to the edge of the page. The concept of function is taught using the common notation $f(x)$. Since so many irrelevant features appear often in concepts, researchers need to be aware of their impact on the learner. The frequency of irrelevant features is a variable that needs to be considered in the study of negative instances.

The geometric task did introduce some new problems that were not found in the 1983 letter-string task. A number of students were unable to identify right angles in treatments with positive and negative instances. One reason might be that students have not been exposed to orthogonal line segments in different orientations. Students who received treatments with all positive instances were not required to recognize line segments that were not perpendicular. It may be that asking students to recognize perpendicular line segments by giving them all positive instances does not evaluate their knowledge of perpendicular. A posttest requiring all subjects to identify positive instances of the altitude of a triangle, would provide a comparison of subjects in the all positive sequence treatment with subjects in the mixed sequence treatment.

Further extensions are necessary before generalizations are made about learning in the classroom. If this line of research continues with similar results, it suggests that negative instances may be helpful in learning the relevant features of a concept when there are irrelevant features that appear frequently.

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PREFACE

The International Group for the Psychology of Mathematics Education (PME) was founded in 1976 at the third International Congress for Mathematics Education. The North American Chapter (PME-NA) was founded in 1979. Like its parent organization, the PME-NA sponsors an annual research conference bringing together psychologists, mathematics educators, mathematicians, philosophers, computer scientists, all interested in problems related to the learning and teaching of mathematics.

This year we have selected as a theme for the plenary sessions "Research in Mathematics Education from an Epistemological Perspective". Traditionnally epistemology is perceived as a branch of philosophy dealing with questions such as "What does it mean to know?", "How does knowledge develop?", "What are the conditions needed for its growth?". These are questions which seem fundamental for the science of education in general. But why choose it as a theme for a colloquium on research in mathematics education?

There are indications that the time is ripe to question the implicit philosophy underlying our past research. It seems that a significant part of the research community is now aware of the dead-end often achieved by a naive behaviorist approach towards mathematics education. For instance, such an approach has led to an algorithmic perception of mathematics, has viewed learning of mathematics as merely the acquisition of skills, and has considered teaching purely as a transmission of knowledge. On the other hand, a constructivist perspective stresses thinking and reasoning in the learning of mathematics, the acquisition of knowledge being perceived as a construction by the learner, and the teacher's role that of guiding him in his construction.

Two such opposite and diverging perspectives necessarily affect the kind of research questions one chooses and the methodologies one uses. The implications need to be discussed and debated thoroughly if we want to avoid two possible pitfalls, that of a researcher spending all his time trying to "solve rigorously" problems devoid of any pedago-

gical significance, or on the other hand trying to take into account so many variables that no possible experimentation can be construed.

In order to help open the debate, two plenary sessions and a panel discussion have been planned. The remaining papers have been grouped under the usual headings. To bring out the bilingual character of Canada, the texts of the two plenary papers and the two reaction papers appear in both English and French. In addition all English communications offer a French abstract and vice versa.

Jacques C. Bergeron
Nicolas Herscovics

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PREFACE

Le groupe international PME (International Group for the Psychology of Mathematics Education) est né en 1976, lors du troisième Congrès International pour la didactique de la Mathématique (International Congress for Mathematics Education). Le chapitre nord-américain (PME-NA) a été fondé en 1979. A l'instar de l'organisme-mère, le PME-NA organise chaque année une conférence de recherche regroupant des psychologues, des formateurs de maîtres en didactique de la mathématique, des mathématiciens, des philosophes, des informaticiens, tous intéressés aux problèmes reliés à l'apprentissage et à l'enseignement de la mathématique.

Cette année nous avons choisi comme thème, pour les séances plénières, "La recherche en didactique de la mathématique dans une perspective épistémologique". Selon la tradition, l'épistémologie est perçue comme une branche de la philosophie traitant de questions telles que "Quel est le sens de connaître?", "Comment la connaissance se développe-t-elle?", "Quelles sont les conditions nécessaires à sa croissance?". Ce sont là des questions qui semblent fondamentales pour les sciences de l'Éducation en général. Mais pourquoi en faire le thème d'un colloque dédié à la recherche en didactique de la mathématique?

Il y a des indications que le temps est propice pour remettre en question la philosophie implicite sous-jacente à notre recherche passée. Il semble qu'un nombre assez important de chercheurs soient maintenant conscients du cul-de-sac dans lequel nous a souvent conduits une approche behavioriste naïve envers la didactique de la mathématique. Par exemple, une telle approche a mené vers une perception algorithmique de la mathématique, a réduit l'apprentissage de celle-ci à l'acquisition d'habiletés, et a considéré l'enseignement uniquement en termes d'une simple transmission de connaissances. Par contre, une perspective constructiviste met l'accent sur la pensée et le raisonnement dans l'apprentissage de la mathématique, l'acqui-

sition de la connaissance étant perçue comme une construction par l'apprenant, et le rôle de l'enseignant comme étant celui de guider l'élève dans sa construction.

Deux telles perspectives opposées et divergentes influencent nécessairement le type de questions de recherche que l'on se pose et le type de méthodologie que l'on adopte. Les implications qui en découlent ont besoin d'être discutées et débattues à fond, si nous voulons éviter deux des dangers possibles, soit celui du chercheur qui passe tout son temps à essayer de "résoudre de façon rigoureuse" des problèmes n'ayant aucune signification pédagogique, ou bien qui essaie de tenir compte de tant de variables qu'aucune expérimentation possible ne peut être envisagée.

Jacques C. Bergeron

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G. MATHEMATICAL ABILITIES / HABLETES MATHEMATIQUES

PERSONALITY DIFFERENCES OF MATHEMATICS
AND ENGLISH PERFORMERS IN HIGH SCHOOL
Martin Cooper, University of New South Wales

RÉSUMÉ

Le but de cette recherche était d'explorer, chez des étudiants de onzième année, les différences de personnalité qui existent entre ceux qui réussissent bien en Anglais et ceux qui sont davantage tournés vers les Mathématiques. Le questionnaire de Cattell "High School Personality Questionnaire" a été administré à plus de quatre cents étudiants dont les résultats en Anglais et en Mathématiques étaient disponibles.

Des analyses univariées exploratoires ont montré que les facteurs G, O, Q2 et Q3 différencient ceux qui réussissent bien de ceux qui ont moins de succès, à la fois en Anglais et en Mathématiques, ceux qui réussissent bien s'avérant plus persévérants, plus consciencieux, ayant une meilleure maîtrise d'eux-mêmes et plus d'autonomie que ceux obtenant des résultats médiocres. Ceux qui réussissent bien en Anglais ont obtenu des résultats plus élevés pour le facteur I que ceux qui arrivent moins bien dans ce domaine alors que les bons mathématiciens ont obtenu de meilleurs résultats pour les facteurs A et C que les mathématiciens faibles; aucun de ces trois facteurs n'a permis d'établir de discrimination dans l'autre discipline.

Le groupe des gens orientés vers l'Anglais et celui des gens tournés vers les Mathématiques ont ensuite été comparés par une analyse multivariée portant sur les facteurs A, C et I; chacun des groupes était constitué des élèves appartenant à la tranche des premiers 40% dans l'une des discipline et des derniers 40% dans l'autre. Seul le facteur I a permis de différencier les groupes de façon significative, les groupe des gens tournés vers l'Anglais s'avérant plus exigeant au plan émotionnel, plus sentimental et plus centré sur la personne.

BACKGROUND

Even when the measurement of personality was in its infancy, researchers were comparing the personality characteristics of groups of individuals having different vocational orientations.

Lynn (1959), for instance, using Eysenck's *Maudsley Personality Inventory*, found that male apprentices and female occupational therapists were both less Neurotic and more Extrovert than first-year university undergraduates of the same age and sex. In a later study, Roe and Siegelman (1964), using Cattell's *Sixteen Personality Factor Questionnaire* (16PFQ), collected data which demonstrated that men committed to engineering as a lifetime occupation were more inflexible, cautious, taciturn, susceptible to threat and self sufficient than those committed to social work.

In the handbook to the 16PFQ, Cattell *et al* (1970) present personality profiles for a number of occupations. Those for writers and chemists differ markedly, chemists scoring substantially lower on factors measuring dominance, sensitivity and imaginativeness, and higher on factors measuring perseverance and inhibition.

The above findings lead one to suspect that persons performing well in subjects which are oriented towards theories and facts have different personality patterns from those making high marks in subjects focussing on people and their intercommunication.

METHOD

The *High School Personality Questionnaire* (HSPQ) (Cattell *et al.*, 1969) was administered to 414 grade eleven students in four high schools. The primary personality traits measured by this instrument are described in Table 1. Mid-year and final marks in English and Mathematics were collected also, the averages being used as measures of performance in English and Mathematics, respectively.

Two sets of analyses were carried out: one comparing high and low performers in each subject separately, the other comparing high-English-low-Mathematics students with their opposite numbers. In the first set of analyses, high and low performance groups were established for English and for Mathematics. These groups, comprising the top-performing and bottom-performing quarter in each subject, were compared separately on each of the factors measured by the HSPQ. For the second analysis, an English-oriented group and a Mathematics-oriented group were formed; each of these groups was made up of students in the top 40% of one subject and the lower 40% of the other. These groups were compared in a multivariate analysis of variance, the HSPQ factors A, C and I being the dependent variables.

Code	Title of Trait	
	<i>high score pole</i>	<i>low score pole</i>
A	Affectothymia	Sizothymia
B	General Intelligence	Mental Defect
C	Ego Strength	Ego Weakness
*D	Excitability	Phlegmatic Temperament
E	Dominance	Submissiveness
F	Surgency	Desurgency
G	Superego Strength	Superego Weakness
H	Parmia	Thretica
I	Premia	Harria
*J	Asthenia	Zeppia
#L	Protension	Alaxia
#M	Autia	Practical Concernedness
#N	Sophistication	Naivete
O	Guilt-proness	Confident Adequacy
#Q1	Radicalism	Conservatism
Q2	Self-sufficiency	Group Dependency
Q3	Strong Self-sentiment	Weak Self-sentiment
Q4	High Ergic Tension	Low Ergic Tension
* included in HSPQ but not in 16PFQ		
# included in 16PFQ but not in HSPQ		

Table 1
The Primary Source Traits Measured by the 16PFQ and the HSPQ

RESULTS

The first set of analyses constituted univariate F tests comparing high-performers with low-performers in English and Mathematics separately. Mean scores on the respective personality factors measured by the HSPQ for high and low performance groups in English and Mathematics are shown in Table 2.

The values of F which were significant at the 0.01 level or better are indicated below. (The asterisks indicate significance at the 0.05 level.)

HSPQ factor	English		Mathematics	
	F	P-value	F	P-value
A			7.88	0.0055
C			7.42	0.0070
G	25.64	0.0000	20.66	0.0000
I	8.98	0.0031		
O	7.47	0.0069	*	
Q2	*		18.37	0.0000
Q3	13.86	0.0003	12.08	0.0006
numbers of deg. of frdm.	1 and 198		1 and 197	

Factors G and Q3 differentiate good-performers from poor-performers for both English and Mathematics. Factors O and Q2 differentiate good performers from poor performers in one subject at the 0.01 level and in the other subject at the 0.05 level. Factors A and C discriminate good performers from poor performers in Mathematics but not in English, and Factor I differentiates between good and poor performers in English but not Mathematics.

	English		Mathematics	
	low performance group	high performance group	low performance group	high performance group
A	11.16	11.39	11.89	10.49
C	9.87	10.62	9.66	11.10
D	9.64	9.46	9.76	9.07
E	10.04	9.26	9.07	9.85
F	10.29	9.96	10.48	9.36
G	10.24	12.44	10.29	12.33
H	10.18	10.11	10.47	9.95
I	8.55	10.95	10.18	9.89
J	8.24	8.69	8.56	8.64
O	8.83	7.50	8.70	7.62
Q2	8.74	9.98	8.40	10.52
Q3	10.01	11.85	10.18	11.77
Q4	9.54	9.56	9.24	9.09

Table 2. Mean scores on HSPQ factors for high and low performance groups in English and Mathematics

The means set out in Table 2 indicate that those doing well in English tend to make high scores on Factors G, I, Q2 and Q3 and low scores in Factor O. High performers in Mathematics tend to score well on Factors C, G, Q2 and Q3 and make low scores on Factors A and O.

Given these results, the question arises as to whether Factors A, C and I discriminate jointly between English-oriented and Mathematics-oriented students. A multivariate analysis of variance comparing these groups on Factors A, C and I gave $\Lambda = 0.802$ with arguments 3, 1 and 57. Since the 99th centile of Wall's U with these arguments is 0.814, the Λ -value is significant at the 0.01 level. (The P-value associated with the corresponding

values of the Roy and Bartlett criteria is 0.0065.)

Post hoc Wilks confidence intervals with $\alpha = 0.01$ indicated significance in the case of Factor I and non-significance for each of Factors A and C. The mean score on Factor I was higher for English-oriented students than for their Mathematics-oriented peers.

DISCUSSION

The results indicate that the set of personality characteristics which differentiates good performers in English from poor performers is different from the corresponding set for Mathematics.

Those doing well in grade-eleven English are more persevering, conscientious, self-controlled, self-sufficient, emotionally demanding and person-oriented than those who do poorly.

High performers in grade-eleven Mathematics are also more persevering, conscientious, self-controlled and self-sufficient than those who do poorly but, instead of being more emotionally demanding and person-oriented, they are more aloof, reserved, and able to control their emotions.

Factor I differentiates students who perform well in English but poorly in Mathematics from those who perform well in Mathematics but poorly in English. English-oriented students are more emotionally demanding, sentimental and person-oriented than their Mathematics-oriented colleagues.

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REANALYSES OF FACTOR-ANALYTIC STUDIES OF MATHEMATICAL ABILITIES

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RESUME

Des analyses nouvelles de certaines des études, parmi les mieux conçues méthodologiquement, portant sur l'analyse factorielle des habiletés mathématiques ont été entreprises afin de clarifier les "patterns" dans la synthèse des résultats originaux (DeGuire, 1982-1983) et d'explorer, par les facteurs de premier ordre, la structure hiérarchique des habiletés mathématiques. Douze ensembles de données ont été analysées de nouveau, ces données provenant des chercheurs suivants: Barakat (1951), Campbell (1956), Kline (1959), Mitchell (1938), Very (1967), Weiss (1955), Werdelin (1958) et Wrigley (1958). Les résultats de ces nouvelles analyses ont été organisés selon les mêmes familles de facteurs issues de la synthèse de la masse globale des données et ont été alors comparés les uns aux autres ainsi qu'aux résultats originaux.

La même diversité, qui parmi les études choisies et parmi les nouvelles analyses constituait un obstacle à la synthèse statistique des résultats, vient étayer ici la possibilité de généralisation des résultats concernant la structure. Dans l'ensemble, les résultats portent à conclure que les habiletés de raisonnement et les habiletés représentées par l'intelligence à la fois fluide et cristallisée sont étroitement liées à la réussite en mathématiques, que les habiletés spatiales et numériques sont pour leur part liées à certains aspects de la réussite en mathématiques et que les habiletés verbales ne sont pas associées de très près à cette réussite. La structure partielle des habiletés mathématiques qui émerge de l'étude représente, avant tout la relation qui existe entre les habiletés mathématiques et les habiletés cognitives plus générales.

Factor analysis has been used throughout this century to study the structure of mathematical abilities. DeGuire (1982, 1983) selected 48 such studies (hereafter referred to as the pool) and conceptually synthesized their results. The factors were compared in six families: General, Numerical, Reasoning, Spatial, Verbal, and Mathematical. The search for patterns in these results was hindered by deficiencies in some of the analyses, by widely varying factor-analytic techniques, and by the lack of attention to higher-order factors. Thus, reanalyses of some of the better designed studies in the pool were undertaken in order to clarify patterns in the synthesis of the original results and to look for a hierarchical structure of mathematical abilities through higher-order factors.

PROCEDURE

In each reanalysis, the original correlation matrix was subjected to a principal factor analysis with communality estimates in the diagonal of the matrix, followed by graphical rotations to oblique, simple structure. Higher-order analyses followed the same procedure. The entire structure was orthogonalized by the Schmid-Leiman (1957) procedure and the orthogonalized factors interpreted. The reanalyzed results were organized into the same families as in the synthesis of the pool, and were then compared to each other and to the original results. Percents of total variance accounted for by factors in the family were also calculated and used to examine the association between the family and mathematical abilities.

THE REANALYSES

The following twelve data sets from studies in the pool were reanalyzed: Barakat's (1951) female and male, Campbell's (1956), Canisia's (1962), Kline's (1959) study B, Mitchell's (1938), Very's (1967) female and male, Weiss's (1955), Werdelin's (1958) Alpha study, and two of Wrigley's (1958) data sets. Details of each reanalysis are presented in DeGuire (1983, pp. 121-180, 247-314). The Canisia reanalysis was terminated due to technical problems in the extraction of first-order factors. The number of students in the

samples for the reanalyses totaled more than 2100, approximately 1700 male and 400 female. The majority of the students were 13, 14, or 15 years of age. The samples were from four countries: United States, England, Northern Ireland, and Sweden.

COMPARISON OF RESULTS ACROSS REANALYSES

The General Family. In the reanalyses, more, and more clearly defined, second-order factors were extracted than in the pool. Also, third-order factors were extracted in the reanalyses, though most of them were vague. Cattell's (1971) higher-stratum functions in his triadic theory of ability structure were used as a basis for comparing the higher-order factors in the reanalyses. Most identifications of higher-order factors were made with fluid intelligence (Gf) and crystallized intelligence (Gc). The other higher-stratum functions were too underrepresented to suggest any patterns. Nine second-order factors in the reanalyses could be roughly identified with fluid intelligence (Gf). The percents of total variance accounted for by the factors had a mean of 8.5% with a standard deviation of 3.4%. Only five of the nine percents of total variance fell within one standard deviation of the mean. The loadings of mathematical first-order factors on second-order Gf-like factors suggested that mathematics achievement is significantly associated with individual differences in fluid intelligence. Eight second-order factors in the reanalyses could be roughly identified with crystallized intelligence (Gc). The percents of total variance accounted for by the factors had a mean of 9.2%, with a standard deviation of 5.6%. Only four of the eight percents of total variance fell within one standard deviation of the mean. The loadings of mathematical first-order factors on second-order Gc-like factors suggested that mathematics achievement is significantly associated with crystallized intelligence.

The Numerical Family. Most of the factors in the Numerical family in the reanalyses were closely related to the standard Number factor, the standard Perceptual Speed factor, or both. The percents of total variance accounted for by the Numerical family had a mean of 6.6% and

a standard deviation of 3.2%. In all but two of the reanalyses, these percents fell within one standard deviation of the mean. The loadings of mathematical measures on factors in the Numerical family followed patterns similar to those found in the synthesis of the pool, but also suggested certain sex differences in factors in the Numerical family. For females, individual differences in the automatization represented by the Numerical family were associated with performance on complex computation tests and algebraic tests.

The Reasoning Family. The most common kind of reasoning factor in the reanalyses was Inductive Reasoning (I). The percents of total variance accounted for by the Reasoning family had a mean of 6.9% and a standard deviation of 5.0%. In only five of the reanalyses did these percents fall within one standard deviation of the mean. The loadings of mathematical tests on the factors in the Reasoning family did not clarify the relation between reasoning and mathematical abilities. Numerical measures, complex computation measures, and algebraic measures loaded on some reasoning factor in most of the reanalyses in which they were included, but did not load consistently with one kind of reasoning factor. No geometric measure loaded on a reasoning factor in the reanalyses, even though such tests were included in the batteries of five of the reanalyses. The geometric measures all loaded on Mathematical factors.

The Spatial Family. Most of the Spatial factors in the reanalyses exhibited more than one aspect (spatial orientation or visualization) and about half included reasoning aspects. The percents of total variance accounted for by the Spatial family had a mean of 4.8% and a standard deviation of 2.2%. In all but two of the reanalyses, these percents fell within one standard deviation of the mean. The loadings of mathematical tests on spatial factors in the reanalyses supported the conclusion in the synthesis of the pool that certain spatial abilities are associated with the performance of certain geometric tasks.

The Verbal Family. In only eight of the reanalyses were factors identified that have been classified in the Verbal family. The percents of total variance accounted for by the Verbal Family had a mean of 5% and a standard deviation of 3.6%. In only four of the eleven reanalyses did these percents fall within one standard deviation of the mean. Few mathematical tests loaded on the factors in the Verbal family. Those that did were mathematical vocabulary or reading tests or tests with verbally-presented tasks.

The Mathematical Family. Most of the Mathematical factors in the reanalyses closely paralleled factors in the original results. Consequently, they led to the same conclusions as for the Mathematical family in the entire pool, that is, that the relationship of achievement between and within branches of mathematics clearly depends upon the nature of the measures included in the analysis. Further, the results of the reanalyses strongly supported Mathematical factors as a kind of factor, though they did not establish any particular Mathematical factor as stable. The Mathematical factors in the reanalyses that did not parallel factors in the original results suggested some interesting observations. The Mathematical Expression Knowledge factor in the Weiss reanalysis seemed to be a mathematical equivalent to a Word Knowledge factor in general vocabulary. Apparently, not all the abilities utilized in mathematical vocabulary tests are the same as abilities utilized in general vocabulary tests. Differences in these abilities have not been identified or explored in the studies in the pool. Mathematical factors in the Kline reanalysis exhibited an interesting distinction among mathematical factors within the same branch of mathematics. They were distinguished by the level of understanding required rather than by the topics within the tasks. Other studies of a single branch of mathematics have identified very specific and content-distinguished factors.

Scope of Abilities Represented. Overall, the percent of variance accounted for by the hierarchical factor structures had a mean of

54.9% and a standard deviation of 6.6%. For eight of the eleven reanalyses, these total percents fell within one standard deviation of the mean. Given the diversity of samples and test batteries in the reanalyses, this similarity is interesting but its meaning is not clear. Certainly, some fraction of the remaining variance can be attributed to error of measurement and factors irrelevant to mathematics achievement. Test reliabilities in the batteries of the reanalyses were generally 80 to 90%, indicating that 64 to 81% would be a ceiling for the percent of variance accounted for by the factor structures. So a fraction of the variance is still unaccounted for. This result leaves open the possibility that non-achievement abilities exist that are specific to mathematics.

CONCLUSION

The same diversity among the studies in the pool and among the reanalyses that prevented a statistical synthesis of the results strengthens the generalizability of the finding concerning structure. Overall, the results suggested the conclusions that reasoning abilities and abilities represented by both fluid and crystallized intelligence are closely associated with mathematics achievement, that numerical and spatial abilities are associated with certain aspects of mathematics achievement, and that verbal abilities are not closely associated with mathematics achievement. Factors in the Numerical and Spatial families accounted for about 7% and 5%, respectively, of the variance in mathematical abilities as measured by the batteries in the reanalyses. The percents of variance accounted for by the Reasoning, Verbal, or Mathematical families were less definite. Fluid- and crystallized-intelligence-like factors each accounted for approximately 9% of the variance. The group of higher-order factors and the group of first-order factors each accounted for approximately 25% of the variance.

Figure 1 shows the mapping between general abilities and mathematical abilities suggested by the results. The small squares within the set of mathematical abilities symbolize the many aspects of mathematical

abilities, with those squares inside the small oval representing the abilities examined in this study. The small squares within the set of mathematical abilities but outside the small oval are included to leave open the possibility that mathematical abilities are not merely a subset of general cognitive abilities.

The partial structure of mathematical abilities that emerged in this study primarily represents the relationship between mathematical abilities and general cognitive abilities. Few of the results in the selected studies offered insights into the structure of abilities specific to mathematics, though their existence was confirmed. Future research needs to extend the structure of mathematical abilities. Future research might also vary the definition of mathematical abilities. In the pool, mathematical abilities were operationally defined implicitly or explicitly in terms of the abilities to do school mathematics. One can, however, define mathematical abilities in terms of the abilities to learn rather than to do school mathematics. Mathematical abilities can also be defined other ways, for example, in terms of the abilities to create mathematics. When such expansions of the definition of mathematical abilities are considered, one sees that only a small part of mathematical abilities has been investigated.

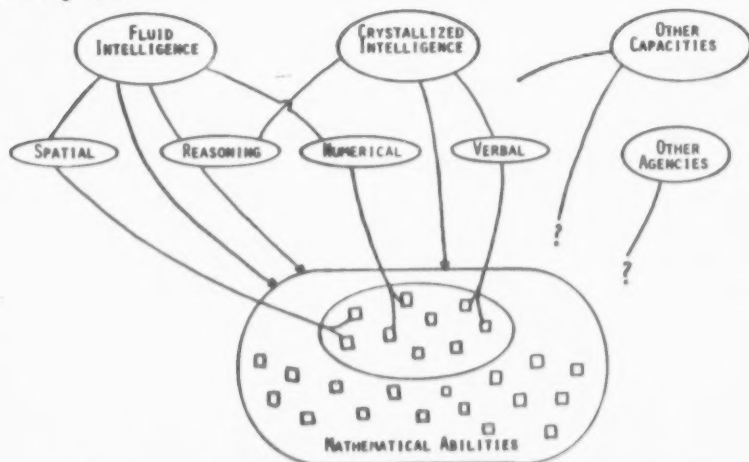


Figure 1. Mapping between General and Mathematical Abilities

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H. PEDAGOGICAL PROBLEMS IN THE TEACHING OF MATHEMATICS
AT THE UNIVERSITY LEVEL /

PROBLEMES PEDAGOGIQUES DANS L'ENSEIGNEMENT DES
MATHEMATIQUES AU NIVEAU UNIVERSITAIRE

ON SOME PROBLEMS OF TEACHING PROBABILITY THEORY

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RESUME

De nos jours, la théorie des probabilités constitue un chapitre important des mathématiques modernes, pures et appliquées. Une part considérable de la recherche appliquée, qu'elle soit ou non reliée aux sciences naturelles, au génie, à l'industrie, à la médecine ou à l'économie, est largement influencée au niveau de ses processus sous-jacents, par des phénomènes ou des perturbations aléatoires. Ceci a, sans doute possible, constitué un stimulus puissant pour l'accélération du développement de la théorie actuelle des probabilités et des statistiques mathématiques.

Aujourd'hui il n'est aucun Collège, aucune Université qui n'ait de programmes dans ces domaines. Des efforts constants visent la création de nouveaux programmes établissant des liens entre d'une part, la théorie des probabilités et les statistiques et, d'autre part, de nouvelles voies en sciences naturelles, génie et sciences humaines. Avec l'établissement de nouveaux programmes et de nouveaux cours, il est normal que des problèmes apparaissent. Dans ce texte, nous voulons étudier un aspect de ces problèmes, à savoir le problème de l'enseignement des probabilités dans les Collèges et Universités.

The theory of probability in our day is an important chapter in both theoretical and applied mathematics. A significant part of applied research, whether it applies to the natural sciences, engineering, the organization of production, medicine or economics, is based on the essential fact that actual processes are subject to random influences. Undoubtedly, this general conception was a powerful stimulus for the rapid progress in the theory of probability and in mathematical statistics.

The emergence of the computer in the last three decades and its effects on other sciences has posed a multitude of problems which were solved by applying models of probability theory. Many complex problems concerning the stochastic behaviour of computer systems invoke the theory of stochastic processes in their solutions.

Today, there is no college or university which has no programs in probability theory and mathematical statistics. Efforts are constantly devoted to initiate new programs which link this field to new directions in natural sciences, engineering and human sciences. With the establishment of new programs and new courses, it is quite natural that problems arise.

In this paper, we will investigate one aspect of these problems, namely the problem of teaching probability and mathematical statistics in colleges or universities. In this work we will concentrate on one aspect of the posed problem namely the mathematical preparation of students entering colleges or universities. It is well known that the entire elementary and secondary school education in mathematics is based fundamentally on deterministic mathematics. Geometry, algebra, numbers are sufficiently covered in the school curricula. Recently some aspects of set theory started to emerge. Combinatorial aspects of mathematics have been almost ignored but the science of randomness, stochastic phenomena have been completely ignored and the majority of students entering university have a great difficulty when faced with such notions as random events,

sample spaces, frequency distribution, etc.

People meet at random, they marry each other at random. The conception of a child, his birth time are random. The lifetime of a human being or a machine is random. All aspects of life surrounding us are random, yet our school curricula are not reflecting these important phenomena. Random phenomena occur in physics, biology, chemistry, economics, medicine, etc.

The fundamental goals of secondary education are well known; to give students the basis of the scientific world view, to acquaint them with the basic conclusions of modern science, and to prepare them for independent work. All the above goals are so important that they will always remain at the basis of education. The content and the methods of education, however, will change. These changes are unavoidable, since with time, societal needs change, new professions come into being, while others become obsolete; the volume and depth of stored knowledge grows and forms the basis for changes in scientific concepts, as well as in industry. Consequently, both the goals and content of education must also change. This is why the timely review of current and fundamental changes is so necessary. This will allow for the opportunity to consider society's needs more closely and more rapidly, while widening the content of education with due consideration of all pedagogical requisites.

At the present time, the need has arisen to introduce elements of the theory of probability and statistics into the school curriculum. The same needs arise in the methodology of school teaching and in the students' further practical activities, in interdisciplinary relationships, primarily in physics, biology and chemistry. Currently, molecular and submolecular conceptions of the structure of matter are very important in modern physics. Since the first quarter of the past century, statistical methods have been introduced into biology, and now, they are the basic and only mathematical apparatus of genetics, elements of which are taught in school.

Many notions in modern chemistry, such as the notion of equilibrium, require an understanding of the theory of probability. Furthermore, through the theory of errors in measurement and through the design of experiments, statistical concepts have left a deep imprint on the conscience of those involved in experimental work, in precise testing, and experimentation.

The twentieth century has seen the preeminence of statistical conceptions in all the natural sciences: in economics, in industry and in engineering. The situation is such that many people, including those who are not involved in scientific research, require elements of statistical knowledge to develop a wider range of understanding of the laws of nature, social phenomena, and technological processes than those developed by mankind during thousands of years, and which have been vividly reflected in the mechanical determinism which dominates education at the present time.

We cannot disregard the accidental, wave off stochastic occurrences or consider them undeserving of scientific attention, since they constantly surround us and pervasively intrude into our lives. The student must gain an understanding of random phenomena from mathematics and physics, from botany and zoology, chemistry and literature. And this introduction to the world of stochastic phenomena will be of great benefit to his future life, irrespective of what he chooses to become - a worker, an engineer, an office worker, or a scientist.

Today, science has discovered the causes of random occurrence or stochastic phenomenon, for these also have their own causes and are governed by particular laws which include as a particular case (more precisely as a limiting case), the laws of firmly deterministic phenomena. In this way mankind has attained a wider understanding of the laws and determinism of phenomena in comparison with the deterministic thinking typical of the XVIII and the beginning of the XIX Century. There is no need to repeat that the school population

should not be kept in the dark and uninformed about the basic information on the laws of random phenomena or about the primary methods of quantitative study which are already developed in science.

Since molecular structure of matter won recognition in physics, statistical concepts have become unavoidable. The enormous number of particles in constant and chaotic motion can not be studied by the usual methods of purely deterministic analysis, if only for the reason that they are so many in number and we do not know the form of these particles nor the forces directing their interaction. We have at our disposal, only a little general information: the particles are numerous, they are in a state of chaotic motion, and for the understanding of the state of the whole mass of particles, there is no need to know the individual motion of each particle.

The need arises to determine some of the general laws governing the motion of enormous masses of molecules, based solely on the assumption that there are a great many of them and that the influence of each particle on the whole is not significant. Following this route, and using the simplest concepts of probability theory, it was possible to discover many laws governing mass phenomena similar to the motion of molecules. For instance, some of the laws of nature discovered and known to every school boy, are Boyle-Mariott's law; Gay-Lussac's law, and others. Furthermore, it has become apparent that these laws can not be characterized as absolute truths, since observations have shown that with the reduction in the number of particles involved in chaotic motion, deviations from these laws have been observed. At the same time, the methods of the theory of probability allow for the prediction of these deviations, including the prediction of the frequency of their occurrence.

Biologists from the time of the Belgian biologist A. Quételet (1796-1874) have noted that the variations in the size of organs of living beings of the same type and age fully correspond with the general laws of probability theory. The famous laws of the Austrian researcher G. Mendel (1822-1884) which have laid the foundation of

contemporary genetics, require the application of probability theory concepts in order to understand them and to formulate them quantitatively; and, what is most important, Mendel's theories require the refutation of classical conceptions of complete determinism. Attempts by biologists to ignore the laws of probability theory have, in the recent past, led to the distortion of the laws of nature and to serious consequences for the development of science, as well as for the practical work of agronomist and those engaged in animal husbandry and meat production.

In studying biological phenomena, one constantly comes into contact with the analysis of such characteristics as height, weight, life expectancy, and to capabilities, when analyzing humans or animals. A student familiar with the laws governing stochastic phenomena at least superficially in his school years will be prepared psychologically when entering a course in probability theory or mathematical statistics. He will probably enter such a course with full understanding of its importance as well as its power to resolve many practical problems.

In conclusion we have touched on only one aspect of the problem namely the mathematical preparation of students entering courses in probability and mathematical statistics. However, we did not touch other aspects such as methodological problems, teacher preparation, teacher's exposition of material; historical overview of material offered which are so useful in drawing the listener's attention to the teacher's lesson. In a forthcoming paper we treat all or some of these points in full details.

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HISTORICAL AND PEDAGOGICAL ASPECTS OF THE CONCEPT OF LIMIT

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RESUME

Un des grands défis que doit relever le professeur de mathématiques du premier cycle universitaire a trait à l'enseignement du concept d'une limite. Et quoique l'on ait discuté de cette question depuis plus d'un siècle, on ne s'accorde pas sur une manière de présenter cette notion qui soit valable sur le plan pédagogique. Et le débat n'est pas terminé.

L'expérience nous apprend que c'est au fil de ses divers cours de Calcul et d'Analyse que l'étudiant finit par saisir enfin cette notion de limite et que sa compréhension évolue selon un certain "ordre". Dans cette communication nous signalons que cet "ordre" correspond au développement historique du concept de la limite, depuis Newton jusqu'à Cauchy.

In post-secondary mathematics, the most difficult topic to learn as well as to teach is the definition of limit: A is the limit of $f(x)$ as x tends to a if for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \epsilon$ for $|x - a| < \delta$. It is easy to state the definition but its understanding requires mathematical thinking at a level much higher than what student has acquired in school mathematics. Math. teachers undertake the responsibility to communicate this idea to students but do not feel satisfied with the performance of students in understanding this concept. Mostly, the teachers consider it either as their own failure or incompetence of students rather than attributing the blame on the intrinsic difficulty of the $\epsilon - \delta$ definition of limit. Math. educators search for and speak on mathematics curriculum that would provide an appropriate background before the concept of limit is introduced. So far, there seems to be no success.

The concept of limit is the fundamental concept of calculus. It is the very foundation on which the mathematics of calculus is based. This notion has to be introduced in the beginning of the first course on calculus before the chapter on derivative starts. It is one of the paradoxes of pedagogy that such a difficult idea has to be taught at early stage of mathematics study. In fact, the learning of the concept of limit is not a one-shot affair. The correct definition of limit is at hand but the student understands the essence of the definition not exactly the same what he is taught. His understanding evolves through a process. He understands the concept in a certain way, criticize his own understanding, reconstruct his understanding then review it again and so on. There is a process of evolution in arriving at the correct understanding of the concept. It is interesting to observe that this process of evolution very well corresponds to the historical developments of the concept of limit.

In 1667, Newton conceived the concept of derivative and the fundamental theorem of calculus: Let there be a curve and the area under the curve be given by $z = x^{n+1}/(n+1)$. Let there be an infinitesimal

increase σ in x . The new abscissa is $x + \sigma$ and the augmented area is $z + \sigma y = (x + \sigma)^{m+1}/m+1$. Applying binomial theorem, one gets

$$\sigma y = x^m \sigma + \frac{m}{2} x^{m-1} \sigma^2 + \dots$$

Dividing throughout by σ and neglecting the terms still containing σ , the result is $y = x^m$. That is, if the area is given by $z = x^{m+1}/m+1$ the curve will be $y = x^m$. Conversely if the curve is $y = x^m$ the area will be $z = x^{m+1}/m+1$.

Along with this observation, Newton conceived the fundamental theorem of calculus and introduced the concept of derivative. This is the beginning of the concept of limit. The clear cut definition of limit is given by Cauchy in the early 19th century. From Newton to Cauchy, the concept of limit evolved over a period of one and a half century from its inception to perfection.

In the first course on calculus, the $\epsilon - \delta$ definition is taught during the first week of the course and it is immediately followed by the derivative of function. The functions dealt with are easy and the derivative is obtained after some algebraic manipulation by putting $\Delta x = 0$. There is no direct appeal to the definition of limit for obtaining the derivative and when the student develops the skill for solving the problems, the underlying concept of limit is overshadowed by technique. The student understands limit as suggested by the techniques i.e. following an algebraic manipulation put $x = a$, $\Delta x = 0$ what Newton did. Nevertheless, he does question the meaning of: Let there be an increase Δx and then to get the answer put $\Delta x = 0$. His doubt is the same what Berkeley presented when he criticized Newton's definition of the derivative. In 1734, Berkeley pointed out: that Newton had disregarded the law of contradiction is assuming first he has an increment and then in order to reach the result allowing the increment to be zero assuming that there is no increment.

In other criticism advanced by Berkeley he says: the velocity de-

depends on space and time intervals and it is consequently impossible to conceive of any instantaneous velocity i.e. a velocity in which these intervals are zero. While doing the problems on speed and motion, the student also feels uneasy with 'the speed at an instant'. He is also not clear on the relation between derivative and tangent. This is what Berkeley also pointed out: in finding the tangent by means of differential one first assumes an increment; but these determine secant not the tangent. One undoes this error by neglecting differentials of higher order and thus by virtue of two-fold mistake, you arrive, though not at science, yet at the truth.

At this stage, the student relies on variation in geometrical forms in understanding the limit. This is what Robin spoke of a circle as a limit of regular polygonon when the number of sides are indefinitely increased.

As the course progresses on, the student observes that most of the problems dealt with are related to physical problems and so, he develops a notion of limit heavily dependent on geometry and motion. One may note that Maclaurin, in 1742 defended the use of geometry in understanding the limit and considered velocity, motion as well as properties of figures as parts of mathematics. All this time, the student has the correct definition at hand but his perception is tied to geometry and motion. In his conception the limit is one-sided as if the variable is always less than or greater than the limit and could not oscillate. This is also implied when L'Huilier, in 1787 said: Given a variable quantity always smaller or greater than a proposed constant quantity but which can differ from the latter by less than any proposed quantity is called the limit in greatness or smallness of the variable quantity.

The calculus course ends but it does not provide an opportunity to the student where there is a direct appeal to the $\epsilon - \delta$; it does not provide a situation where the concept in its precision is needed. If the student continues further in mathematics and enrolls in

a course on Analysis, he is once more told the same definition of limit. In this course, he finds no mention of velocity and area but of least upper bound, continuous function, intermediate value theorem etc. He comes across to a mathematics independent of time and motion and even of geometry. He is now psychologically prepared to understand and accept the definition of limit. One must note that Bolzano in the early 19th century advanced the idea that the introduction of time and motion in mathematics is unnecessary. Bolzano advocated that mathematics should develop on its own, in response to its inner drive rather than depending on physical world. In fact, this doctrine steers the mind to a training in rigor and in abstract thinking. Only after such an approach was adopted the correct definition of limit was given by Cauchy: When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others.

As a final remark, one may note that the above definition is entirely rhetorical. It was later arithmetized in ϵ - δ notation by Weierstrass: The number L is the limit of a function $f(x)$ at $x = x_0$ if given an arbitrary small number ϵ another number δ can be found such that for all values of x differing from x_0 by less than δ , the values of $f(x)$ will differ by less than ϵ .

The task of using the symbol $<$ (less than) and $|\cdot|$ (absolute value) was completed by later mathematicians. One may observe that the student understands the definition first in descriptive form and then along with the use of symbols as if even at this stage, his understanding follows the historical order.

I. PHILOSOPHY AND PERCEPTION OF MATHEMATICS /

PHILOSOPHIE ET PERCEPTION DES MATHÉMATIQUES

BEYOND STRUCTURE: SOME THOUGHTS ON THE NATURE OF MATHEMATICS

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RESUME

Même si les mathématiques représentent beaucoup plus que de simples structures formelles, les maîtres à tous les niveaux traitent souvent la structure comme l'aspect le plus important du sujet. Plusieurs des problèmes reliés à l'enseignement et à l'apprentissage des mathématiques émergent de ce point de vue particulier. Pour remédier à ce problème, il sera nécessaire de commencer à considérer les mathématiques de manière à incorporer à la fois leurs dimensions structurales et leurs dimensions de processus. Cette nouvelle philosophie devra notamment reconnaître le substrat intuitif des concepts mathématiques puisque c'est de ce champ informel que sont issues les définitions et structures formelles. En tant qu'éducateurs en mathématiques, nous devons tourner notre attention vers cette zone primordiale de l'activité mathématique.

BEYOND STRUCTURE: SOME THOUGHTS ON THE
NATURE OF MATHEMATICS

Due perhaps to the influence of Bourbaki, university mathematics courses in the last twenty years tend to be organized around the study of some particular mathematical structure. Thus many courses will focus on one or more mathematical categories, for example, topological spaces and continuous functions, groups and group homomorphisms, or abstract measure spaces and measure preserving mappings. One will rarely find a course which deals with the real number system as a whole. Rather various properties of the reals are abstracted out and discussed as examples of more general structures. In an algebra course we look at the reals as a field, in topology or analysis as a metric or topological space, and in measure theory we may study the σ -algebra of Borel or Lebesgue measurable sets. In each case we isolate a certain structure out of the richness and complexity of the real number system and use this axiomatic structure as a perspective from which to look back at the real numbers. Of course it should be remembered that each structure automatically isolates a certain class of functions, namely those which preserve the given structure, and these are studied together with the given structure.

Now it is not quite true to say that the idea of category lies behind the organization of high school or elementary school mathematics. However there remains a sense in which "structure" remains the main object of concern. It often seems to me that there is a feeling that some ultimate mathematical reality is embedded in these abstract structures. An extreme view would be that mathematics is structure. This formalist point of view holds that one begins with a set of axioms and definitions and deduces mathematical theorems using the rules of logical inference. Understanding a mathematical procedure or theorem might then mean being able to deduce it from a given set of axioms. Specific examples are then often seen as being merely pale reflections of the world of structure - they have no intrinsic value in themselves. The abstract is more real than the concrete

and the mathematical enterprise consists in making explicit what is implicit in the axiomatic system.

In what follows we shall briefly look at some of the consequences of this formalist/structuralist conception of mathematics. We shall argue that many of the problems associated with the teaching and learning of mathematics follow from this narrow perspective on the nature of the subject. What we claim is needed is a new philosophy of mathematics; one that is more in keeping with what goes on when one actually does mathematics. That is we are searching for a definition of mathematics which will include the formal, deductive aspects of the subjects but will not be limited to them.

Recall Morris Kline's example from "Why Johnny Can't Add" [1].

Teacher: "Why is $2 + 3 = 3 + 2$?"

Student: "Because both equal 5."

Teacher: "No, the correct answer is because the commutative law of addition holds."

Though Kline tends to caricature the attitude of what was once called the "New Math", this particular example makes a useful point. The truth of numerical relations is seen to follow from some field axiom. More generally, operations in arithmetic are often "explained" to children by breaking them into a sequence of steps each of which is justified by a mathematical law or axiom. Here abstract principles and deductive reasoning are used to justify the concrete. The assumption seems to be that the more abstract (and therefore general) a proposition the higher its truth value. Learning, of course, often proceeds in the opposite direction, that is, from the specific to the more general. Thus the commutativity of addition is an interesting fact which children often discover when attempting to add a large number to a small one by counting. It certainly makes no sense to impose it on them a priori as a law to be memorized. In fact it seems to me that the words "commutative law" should not be used until the student has some experience with a number of commutative and non-commutative operations. Probably this will not

happen until he reaches the university. (As an aside, there is a sense in which the commutative law is true because $2 + 3 = 3 + 2$ and not the other way around.)

A disastrous consequence of the ascendancy of the structuralist conception of mathematics has been the implicit down-grading of the value of experience and intuition. I can remember not bothering to copy down specific examples in my notes as an undergraduate student since I felt that these were merely special cases of a more general theory. I have subsequently arrived at a very different point of view, namely, that the heart of mathematics lies in the richness of these specific examples. Yet in many university courses we derive all kinds of deep theorems without ever showing that these theorems apply to any interesting class of examples. In a calculus or analysis class, for example, students rarely have experience of functions which are not rational and may not even know how to sketch anything more complicated than a cubic. The possible complexity of continuous functions is not even hinted at.

This reduction in the value of experience is responsible for one of the saddest consequences of the structural approach to mathematics - the inability of parents to understand the mathematics that their children are learning at school. Parents find that their own experience has absolutely no value in this "new" mathematical world. I have a friend with a graduate degree from Oxford and a Ph.D. who is deeply frustrated over his inability to do anything constructive about his children's difficulties with mathematics. This man certainly has the intellectual ability to deal with the simple high-school mathematics that his children are learning. What then is the problem?

It is not merely that the emphasis has been shifted from the experiential domain to the abstract principles which are claimed to lie behind it. Parents, students, and even teachers have little perceived need for an abstract axiomatic system. At most they must accept

the view of some authority as to the value of such a system. Often elementary school teachers are frightened by this approach and proceed to learn it by rote. This often leads to results which would be amusing if they were not so sad. I can recall one teacher would not accept Venn diagrams in which sets were represented by any figure other than a circle (except the "universal set" which had to be represented by a rectangle).

Even textbook writers develop peculiar ideas. Consider the following: (1) "The two basic operations from a mathematical viewpoint are addition and multiplication. In elementary mathematics both of these are undefined, but subtraction and division are defined" [2], and from another book (2) "Multiplication, like addition is undefined" [3]. What could these statements possibly mean? If addition and multiplication are not defined how do we manage to perform these operations? Does undefined mean that $2+3$ is sometimes equal to 5 and sometimes equal to 8? The authors here are possibly misunderstanding a statement which is sometimes found in the formal definition of some algebraic structure (like a field, for example) to the effect that the operations are "undefined". This refers to the fact that one assumes that nothing is assumed to be known about these operations except that they obey the various field axioms. It does not mean that the addition of integers is undefined!

Such statements reveal a degree of confusion which is truly frightening. Addition, of which the teacher has a deep intuitive knowledge is made into a mysterious "undefined" operation of which he must assume he knows nothing. What is gained by such mystification?

I can recall my reactions as a student to this approach to algebra. Facts which were once intuitively clear (like $a \cdot 0 = 0$) now became suspect. Their truth value was somehow diminished. One's confidence in the validity of the algebraic manipulation which one had successfully performed for years was correspondingly diminished. Many students react by feeling that the world of formal mathematics,

divorced from experience and intuition, is completely arbitrary. It appears to them as a kind of magic which the adept can manipulate towards any predesigned end. In despair they often give up any attempt to make sense of what they are doing and try to get through the material by memorization and rote manipulation.

Many of us were introduced to axiomatics through the study of Euclidean geometry. Unfortunately this subject appears to have been dropped from the high school curriculum (at least in Quebec). However it should be noted that axiomatic euclidean geometry is very different from axiomatized arithmetic and algebra. I do not have the space to go into this difference in detail in these pages but the essential difference to my mind is the strong intuitive, geometric content of Euclidean geometry. Proofs are accompanied at every stage by diagrams and though these latter may not be completely valid they are nonetheless an essential part of the process of arriving at a valid argument. Thus the formalism strengthens the intuitive geometric base. In axiomatized arithmetic or algebra at the elementary or high school level on the other hand a premature emphasis on structure tends to impede the growth of an intuitive feeling for the subject. As we have seen above it may even be destructive of whatever intuition the student may have previously acquired.

Now not only does the currently acceptable philosophy of mathematics contribute to the type of difficulties in teaching and learning which we have described above, but it does not even adequately describe the work of the mathematician. Of course modern mathematics is such a vast domain that it is difficult to make any general statement which will cover the entire field. However it is my feeling that very few mathematicians work as though they were producing random theorems from arbitrary sets of axioms. In many areas there is no feeling that one is investigating a particular category, rather the primary object of investigation is a particular problem or concept. This is not to deny that the idea of mathematical category has been a useful one. There are many ideas and theorems which are best stated in

these terms. The point is that the structure is not the primary idea; if it is of value we use it, if not we search for another helpful perspective.

The above discussion is complicated by the fact that mathematicians often talk about mathematics in terms of various structures. It is a convenient way to divide up mathematics. They also often find it convenient to teach in the usual axiomatic way. Thus unless one has been exposed to research level mathematics, and significant research at that, it is perhaps inevitable to assume that what one has seen is real mathematics. Unfortunately, it is not. What the students is exposed to is a formal, sanitized version of the subject. Over the years every hint of struggle or controversy has been wiped clean. The definitions appear to be the inevitable products of some higher intelligence. The proofs are short and aesthetically pleasing. The mathematical world which is presented to the student is beautiful but completely intimidating. Yet this very student goes on to become a mathematics teacher. He might easily have been exposed to mathematical process, to doing mathematics as opposed to memorizing books full of definitions, and proofs. Yet process is rarely part of the curriculum. Thus he comes to believe a mythologized version of what mathematics is about. This mythology then forms the basis for his own teaching of the subject.

In order to correct these difficulties it is necessary to develop a broader notion of what mathematics is all about. This new notion or philosophy would have to include the process dimension of the subject as well as place the ideas of structure and formalism in their proper perspective. In this short talk I shall only be able to make a few comments and discuss one (important) example.

Fundamental mathematical ideas have a reality which lies behind any specific verbal or symbolic formulation. Concepts such as real number, limit, or continuity serve as organizing centers around which we attempt to order a certain range of mathematical phenomena. Thus a

limit is not that which is defined by the $\epsilon - \delta$ definition but rather that the $\epsilon - \delta$ definition is one way of giving meaning to the idea of "limit". This underlying notion is usually a bit vague and intuitive and the various definitions which we propose are an attempt to pin it down. Historically the idea of limit changed a great deal. The need for a more formal, rigorous definition was associated with the expansion of the definition of function. Thus different definitions of the same concept may be appropriate depending on the range of mathematical objects to which the definition is to apply. During the creative period of a mathematical concept there is often much work which is done in finding the best definition and subsequently in modifying or generalizing it or its realm of applicability.

Consider the idea of continuity. Most people are happy to accept the geometric idea of a continuous function as one whose graph can be drawn without lifting pen from paper. In a first calculus course the student has no need of a more precise definition since the class of functions with which he has any experience is so limited, consisting of polynomials, some rational functions, and a few other analytic functions. The $\epsilon - \delta$ definition is not appropriate here because the student has not been exposed to any situation for which his more elementary intuitive notion of continuity is inadequate. Only when we must deal with functions which are defined by infinite series, for example Fourier Series, does the $\epsilon - \delta$ definition force itself upon us.

Thus there is no "correct" definition of continuity. Different definitions may be appropriate in different contexts. At any rate a deep idea like continuity will never be exhausted by a single definition or theorem. We have managed to turn matters around - no more is the $\epsilon - \delta$ definition identical with the very idea of continuity and specific existences of continuous behaviour mere illustrations of this definition. Now the appropriate definition depends on the range of mathematical phenomena with which we happen to be dealing. We have gained a great deal of freedom by becoming aware of the idea

behind the definitions but we have lost the seeming inevitability of the classical axiomatic system.

This point of view is inconsistent with the structural/formalist philosophy. We are forced to acknowledge a reality which precedes the formalism. A given field of mathematical phenomena may be structured in various ways and therefore we must give equal attention to the process through which the structure is developed. Of course mathematics possesses both a structure and a process dimension. The importance of process in mathematics is ignored by those who would focus on structure alone but both aspects of the subject must be present in the teaching and learning of mathematics. The greatest error in the teaching of mathematics is the failure to bring the process dimension into the foreground. This is difficult to do without a recognition of the background out of which the components of structure emerge.

To many mathematics has the aura of being black or white, right or wrong. There seems to be no room for the ambiguity without which there can be no creativity. This precise aspect of the subject is frightening to some and attractive to others. Those who love the subject most were often attracted by this quality of certainty. However we can now see that this description applies at most to the structural dimension of mathematics (and not even to that if we take Gödel seriously). Within the domain of mathematical process, the "doing" of mathematics by anyone from a young child to a professional mathematician, there is no absolute right or wrong. Here we are free to experiment, to conjecture, to make mistakes and learn from these mistakes. If we can banish the structuralist mythology we may then be able to introduce people to the world of real mathematics from which all too many have been needlessly excluded.

NOTES

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- [3] Teaching the Child Mathematics, C.W. Schminke, N. Maertens, W. Arnold, 1973.

PROCESSUS DE MATHÉMATISATION
ET STRATÉGIES D'ENSEIGNEMENT

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Abstract. We present a method to set up didactic strategies that activate the components of the mathematization process. It is applied to the teaching of elementary enumerative combinatorics. Our study is based on an analysis of this subject matter, and on a typology of cognitive processes which contribute to the meaningfulness of mathematical concepts. These processes are thought from the logical, epistemological, and semiological point of view. A structuration of the space of didactic strategies is given by a systematic hierarchy, and some teaching processes in this space are indicated.

INTRODUCTION

Je présente une méthode d'élaboration de stratégies didactiques qui cherche à activer les principales composantes du processus de mathématisation.

Une "stratégie didactique" est pour moi un plan d'action qui comprend une séquence coordonnée d'activités d'enseignement visant la réalisation d'un objectif spécifique d'apprentissage scolaire. Par "processus de mathématisation", j'entends tout un complexe de processus cognitifs impliquant des entités mathématiques ou prémathématiques.

Pour fixer les idées et concrétiser mon propos, j'ai choisi l'analyse combinatoire comme contenu scolaire. Puisqu'on l'enseigne au Québec surtout au collégial, je me placerai à ce niveau de scolarité (12e ou 13e année).

Mon objectif est d'établir une "combinatoire" de stratégies possibles ou souhaitables appropriées à l'enseignement de la combinatoire énumérative élémentaire à ce niveau. Je la détermine à partir d'une analyse structurale de la matière et d'une typologie des processus cognitifs qui y sont implicables.

1. STRUCTURE DE LA MATIERE

J'ai présenté ailleurs [1] une systématique des configurations combinatoires simples qui recouvre la matière considérée. Un bref résumé suffira ici.

Je répartie les configurations combinatoires simples selon trois modes classificatoires fondamentaux et deux niveaux de formalisation que comprend le tableau suivant:

modes niveaux	fonction	mot	préimage
concret	rangements	arrangements combinaisons	séparations en tas
abstrait	applications	m-uplets	partitions

Au niveau concret, on articule cette classification à partir des rangements simples d'objets dans des cases. De la classe des rangements ordonnés d'objets distincts dans des cases distinctes, on obtient, par oublis successifs des propriétés caractérisant celle-ci, six classes disjointes de rangements. Pour chacune de ces six classes, on spécifie diverses sous classes par l'ajout de propriétés restrictives, par exemple, en imposant que les rangements soient injectifs ou surjectifs. Coordonnant ces deux procédés, on établit ainsi, par variation intramodale, une classification des rangements simples.

On peut, de façon analogue, obtenir une classification des arrangements, des combinaisons et des séparations en tas. Il est aussi possible, par variation intermodale, de relier ces dernières à celle des rangements. Par exemple, si on considère les rangements non ordonnés d'objets (distincts ou non) dans des cases distinctes comme des choix ou sélections de cases, on passe aux arrangements (sélections ordonnées), dans le cas où les objets rangés sont distincts, et aux combinaisons (sélections non ordonnées), dans le cas contraire. On fixe ces diverses interprétations à l'aide de dictionnaires qui traduisent les propriétés caractéristiques des classes considérées. Il va de soi qu'à cette coordination il en correspond une au niveau abstrait des formalisations ensemblistes associées.

On obtient ainsi une véritable carte sémantique du champ de connaissance analysé. En la coordonnant avec les principes de dénombrement (invariance de la cardinalité sous bijection, de la somme, du produit, du berger, inverse du berger, etc.), il devient alors possible de systématiser les méthodes de déduction des formules de dénombre-

ment et, en particulier, d'en dégager les analogies.

2. PROCESSUS COGNITIFS

Je me restreindrai à systémographier la cognition conceptuelle, c'est-à-dire à schématiser la pensée qui conçoit, traite, représente et exprime les idées. Je vais considérer l'univers (fictif) des idéalités conceptuelles de trois points de vue: logique, épistémique et sémiotique. Il y correspondra trois catégories de processus cognitifs objectifs dont je n'identifierai que les principaux noeuds de ramification. Je renvoie à la psychologie pour l'aspect subjectif et concret.

Quelles sont les espèces de concepts qu'identifie la logique? Elle y distingue des concepts génériques et spécifiques qu'elle diversifie essentiellement en objets (individus, classes) et relations. Ces espèces sont d'ailleurs relatives: la généricité peut être partielle et une relation prise comme objet.

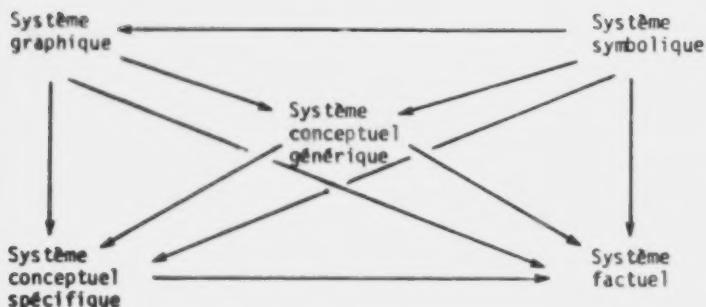
La logique de Port-Royal [2] distingue quatre types généraux d'opérations rationnelles: concevoir, juger, raisonner et ordonner. Il en est ainsi en mathématiques: on détermine des objets, on établit des relations entre ceux-ci, on démontre que certaines de ces relations sont vraies, les théorèmes, qu'on ordonne en une théorie. Ce qui me permet d'épingler une première catégorie de processus cognitifs: les processus discursifs logiques de la définition, de la prédication, de l'inférence et de la systématisation.

Les abordant sous l'aspect méthodologique, on diversifie tour à tour chacun de ces processus. Ce n'est cependant pas mon intention d'établir la typologie des procédures méthodologiques qui s'y raccordent. Je mentionnerai toutefois que les processus de définition par abstraction, de classification, d'induction énumérative et analogique et de formation de contre-exemples sont des plus utiles pour l'enseignement de la combinatoire. On aura compris que je ne considère pas les processus discursifs mentionnés uniquement du point de

vue logico-formel d'une logique de l'exposition, mais surtout du point de vue heuristique d'une logique de la découverte.

La formation de concepts, de propositions, de problèmes et de principes sont des exemples de pensées; il en est de même de leur comparaison et de leur transformation; il en est encore de même de leur intégration en un système. Pour un champ de connaissance qui se constitue, on y perçoit, du point de vue épistémologique, trois types fondamentaux de structuration: l'intra-structuration, l'inter-structuration et la trans-structuration. Ce principe général de structuration se retrouve itérativement à tous les niveaux d'organisation (par méta-structuration), aussi bien en dimension synchrone qu'en dimension diachronique (historique et psychogénétique). J'associe à ce principe divers processus discursifs épistémiques: au premier type de structuration, le processus de différenciation; au deuxième, les processus de mise en relation et de transformation (explicitation des invariants); au troisième, le processus d'intégration.

L'interprétation sémiotique s'applique aux systèmes de signes et aux systèmes conceptuels (considérés comme langages interprétés), va du moins au plus spécifique, peut être partielle ou totale. Elle est définie par une correspondance entre un système source (représentation ou modèle au sens des scientifiques) et un système but (réalisation ou modèle au sens des logiciens). La notion d'interprétation permet de mettre le formel en relation avec les domaines de l'intuition et de l'expérience. Au système interprétatif qui suit, j'associe les processus de représentation symbolique (symbolisation), graphique (schématisation), logico-mathématique (formalisation) et logico ou mathématico-factuelle (modélisation). Les processus de représentation sont en quelque sorte inverses de ceux d'interprétation: linguistiquement parlant, les premiers sont des codages, les seconds des décodages. Rajoutons à ceci le processus de traduction: par transcodage, on relie deux représentations d'un même système ou de systèmes isomorphes.



3. STRATEGIES DIDACTIQUES

Je définirais un processus d'apprentissage didactique par un triplet: un processus cognitif objectif ou de connaissance (commandé par la matière), un processus didactique (initié par le maître), un processus cognitif subjectif ou d'idéation. (sous-tendu par l'élève). J'attribue aux stratégies didactiques la fonction d'induire chez l'élève, par coordination de ses schèmes cognitifs à la structure de la matière, des processus d'idéation homonomes aux processus de connaissance associés.

Une information, une situation concrète, un exemple, une question, une réponse, un exercice, une application, l'évaluation d'une solution peuvent composer diverses situations d'apprentissage. Cependant, puisque les problèmes sont le moteur de la recherche mathématique, ils le restent pour un enseignement axé sur le processus de mathématisation. Dans mon optique, la résolution de problèmes est le centre organisateur autour duquel se ramèfient les activités d'enseignement des mathématiques. Mais la problématique est contrôlée: elle s'ordonne relativement à une théorie qu'elle déploie et alimente. Elle est donc le point d'ancrage auquel est reliée la combinatoire des stratégies didactiques que je propose.

J'en donne un aperçu par l'arbre qui suit:

Stratégie didactique = (Objectif spécifique, Méthode didactique)

Objectif spécifique = (Elément de matière, Processus cognitif)
Elément de matière = (Degré de complexité, Niveau d'abstraction, Hiérarchie de structuration, Modalité conceptuelle, Catégorie conceptuelle spécifique)
Processus cognitif = (Aspect cognitif, Phase cognitive)
Aspect cognitif = (Logique, Epistémique, Sémiotique)
Logique = (Définition, Prédication, Inférence, Systématisation)
Epistémique = (Différentiation, Mise en relation, Transformation, Intégration)
Sémiotique = (Symbolisation, Schématisation, Formalisation, Modélisation, Traduction)
Symbolisation = (Langage naturel, locigo-mathématique, de programmation)
Schématisation = (Figure, Diagramme, Graphe, Tableau, Etc.)
Phase cognitive = (Réception, Elaboration, Emission)
Elaboration = (Analyse, Construction, Synthèse)
Méthode didactique = (Situation d'apprentissage, Forme d'enseignement, Support-véhicule)
Forme d'enseignement = (Forme sociale, Direction de communication)
Forme sociale = (Classe, Equipe, Individu)
Direction de communication = (Maître-élève, Elève-maître, Elève-élève)
Etc.

L'espace des stratégies didactiques, relativement à un champ de connaissance donné, est défini par l'ensemble possible des multiplats extrémaux (les feuilles de l'arbre), et un processus didactique, par un cheminement admissible dans cet espace. J'assigne à la didactique, comme discipline qui se veut scientifique et pratique, la fonction de déterminer et de caractériser les cheminements admissibles optimaux, relativement à certains critères d'apprentissage, dans ces espaces de stratégies didactiques.

Relativement à la structuration des configurations combinatoires simples, trois types de variations se présentent: une variation intraconfigurative (ou intersituationnelle), interconfigurative (ou

intramodale) et transconfigurative (ou intermodale). De la première émerge des classes spécifiques de configurations combinatoires; de la deuxième, des classifications spécifiques; et de la troisième, un complexe classificatoire. Une variation en complexification est d'autre part possible en composant les configurations combinatoires simples par addition ou par multiplication. Finalement, une variation en abstraction est obtenue par la formulation ensembliste.

Une méthode qui diversifie de façon intéressante les processus cognitifs est celle du cheminement inverse. Quelques exemples? Une activité de codage développe l'habileté à représenter des idées mathématiques, tandis qu'une activité de décodage développe l'habileté à interpréter des représentations d'idées mathématiques. Cette dernière fournit d'ailleurs un moyen d'introduire une représentation graphique inconnue des élèves. Si l'élève identifie une configuration combinatoire simple par comparaison ou opposition des traits pertinents des situations décrites par quelques problèmes, il peut aussi l'exemplifier en énonçant un problème. C'est une source merveilleuse de contre-exemples, qui ouvre la voie à d'autres explorations.

Je termine par l'énumération de certaines stratégies didactiques que je trouve importantes pour l'enseignement de la combinatoire: stratégie d'identification d'une classe de configurations, d'identification d'un mode classificatoire, de traduction intermodale, de découverte d'une formule de dénombrement, de découverte d'une formule de récurrence, de recherche d'un principe de dénombrement, de formalisation ensembliste, d'élaboration d'une synthèse. Le temps de mon exposé sera réservé à détailler quelques-unes de ces stratégies.

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COGNITIONS AND THEIR ROLES IN MATH DYSFUNCTIONS

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RESUME

Certaines cognitions sont vues ici comme étant la cause de phénomènes touchant la sensibilité et la cognition et pouvant mener à des difficultés dans l'étude des mathématiques.

Les cognitions nuisibles peuvent être décelées puis corrigées par des moyens pédagogiques.

Research has outlined psychological phenomena which make mathematics and science learning difficult. These phenomena include "attitudes"^{1,2}, learned helplessness^{3,4,5}, perceived utility of mathematics study⁶, and preconceptions which impede learning qualitative concepts in physics^{7,8}. These phenomena can be united by emphasizing the causal roles of cognitions.

COGNITIONS AND THEIR ROLES IN MATH LEARNING

Cognitions are defined naively here, borrowing the definition from cognitive consistency models of attitude change: Cognitions are propositions which can be brought to consciousness, and describe the state of the world, or logical relations. The present discussion concerns cognitions which can affect mathematical behaviors. Examples of such cognitions are "Mathematics is a difficult subject", and "Alternate interior angles are equal."

Second, we propose that cognitions have a causal role in phenomena which affect math learning. Cognitions affect phenomena through two main paths: (a) Cognitions leading to cognitions, which lead to behavior, and (b) cognitions leading to affect, which leads to behavior. These paths are pictured in Figure 1. Let us take each path separately, and describe phenomena which can lead to dysfunction in learners.

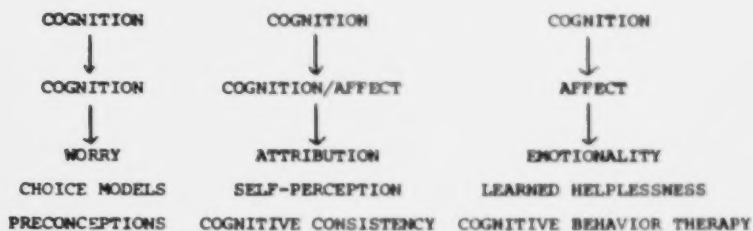


Figure 1

COGNITION - COGNITION - BEHAVIOR PATHWAY

This pathway describes phenomena in which a cognition leads to another cognition which then leads to a behavior. This section describes some phenomena which can be interpreted this way. Preconceptions which are associated with poor performance in physics are examples of such conceptions. Researchers have observed that novice learners hold concepts of kinematics which are less sophisticated than Newtonian concepts of motion. For instance, the "impetus" conception of motion holds that if a body moves with constant velocity, then it is acted upon by a constant force^{8,9}. Similarly, there is a "curvilinear impetus" conception in which an object engaged in curvilinear motion is presumed to have a curved trajectory when released from a constrained orbit¹⁰.

Research also suggests that cognitive factors play a part in individuals' decisions to study mathematics. Rational choice models suggest that individuals' decisions can be made partly upon the basis of cognitive information devoid of affect. Thus, Meece et al.¹¹ show that choices to study mathematics involve many factors other than emotional or attitude variables, such as perceptions of the utility of mathematics in careers, future expectancies, and costs of engaging in specific tasks.

Finally, research in test anxiety suggests that state anxiety has several components, one of which is cognitive, not emotional. Worry is a cognitive aspect of test anxiety which is a concern about performance, and has debilitating effects upon performance¹².

COGNITION - AFFECT - BEHAVIOR PATHWAY

Writers have suggested that mathematics is a troublesome subject because it creates specific attitudes or anxieties of its own^{13,14}. The classical conception of attitudes incorporates the three aspects of cognitive, affective, and behavioral

components. Our present discussion restricts us to phenomena which cognitions precede and cause affect, which leads to behavior.

Learned helplessness is a phenomenon in which individuals acquire a belief that events in the environment are not contingent upon personal actions¹⁵. Dweck¹⁶ shows that students who are judged as learned helpless show deteriorating performance when faced with failure. Learned helplessness is also believed a widespread cause of depression in contemporary models of psychopathology^{17,18}.

Test anxiety research has also uncovered emotionality as a component in state anxiety. Emotionality is a physiological response to anticipating tests or aversive events. However, its occurrence and effects are less predictable than those of cognitive worry^{19,20}.

Cognitive behavior therapies assume that cognition, affect, and behavior are intertwined. The therapies presume that negative or irrational thoughts produce negative affect and self-defeating behavior. Treatment attempts to identify the negative cognitions which produce the affect, and replace them with less distorted cognitions. Proponents of these theories claim success in treating mood disorders such as depression and anxiety^{21,22}.

CAVEAT: THE GRAY AREAS

A number of phenomena are difficult to classify as affective or cognitive, and thus fall in "gray areas". The continuum between cognition and affect is not arbitrary, but a sharp dividing line between the two is arbitrary. For instance, attribution theory describes how individuals infer causal relationships in the world, and interpret their own internal physiological states, thus it seems a part of both "camps". Attitude change theories

of the cognitive consistency or self-perception variety are also difficult to classify. They posit that conflicting or inconsistent cognitions produce unstable internal states that individuals will work to reduce. However, these internal states resemble aversive emotions. Classifying such phenomena probably depends upon classifiers' differences of opinion, and their knowledge of theoretical distinctions.

IMPLICATIONS OF EMPHASIZING THE ROLES OF COGNITIONS IN MATHEMATICS LEARNING

The classification scheme in Figure 1 has several productive implications. First, it serves to unify diverse phenomena in mathematics education by emphasizing that these phenomena are caused by cognitions. The roles of cognitions are clear when analyzing misconceptions in technical thought. However, only recently have contemporary models of psychopathology stressed the causal roles of cognitions in affect and behavior. Thus, understanding how to change cognitions is important in treating both types of phenomena.

The second benefit of considering the roles of cognitions in mathematics disabilities is that it aids assessing and treating troublesome cognitions. Cognitions can be assessed through interviews, paper and pencil tests, and physical models. These techniques have been used to assess misconceptions in physics^{23,24} and may be used to study cognitions in math learning. Paper and pencil tests have also helped identify the worry and emotionality aspects of test anxiety, and to investigate the relations between these factors and task performance.

Once troublesome cognitions are identified, methods of therapy or pedagogy may help modify or replace them. For instance, cognitive behavior therapies employ cognitive restructuring methods to revise beliefs which cause affective disorders. Beck and his associates^{25,26} treat depressive disorders by

identifying and eliminating cognitive distortions by which clients misinterpret events, (e.g., "all or nothing thinking", attributing negative evaluations of themselves to others, etc.). Dweck²⁷ has also demonstrated that attribution theory can help reduce learned helplessness in an instructional setting. Students who exhibited learned helplessness were re-educated to take personal responsibility for performance failures. This teaching increased their resistance to learned helplessness in a mathematics task.

Pedagogy may also succeed in removing inappropriate cognitions through discussion or direct experience. Clement²⁸ remarks that Galileo's new conceptions of motion were difficult to communicate, so Galileo designed Socratic dialogues which would be effective in explaining them. More recently, interactive computer programs have acted as Socratic tutors in teaching students to avoid fallacies and errors in medical diagnosis²⁹. Another pedagogical device is to demonstrate how inappropriate cognitions lead to errors of prediction in mathematics. Carefully designed discussions and exercises may be successful to achieve this goal. LOGO and other rich, interactive computer environments also allow learners to confirm or disconfirm their predictions based upon their understanding of the rules modeled in computer programs.

SUMMARY

- 1) Cognitions are a ubiquitous presence in phenomena which plague math learning.
- 2) The cognitions usually affect behavior through one of two pathways:
 - a) Cognition-Cognition-Behavior
 - b) Cognition-Affect-Behavior
- 3) Many dysfunctional phenomena can be classified into these pathways.
- 4) This classification theme aids theory, for it relates

divergent research in math education.

5) This scheme also holds practical promise for pedagogy, for cognitions may be changed through some of the following methods: a) Cognitive restructuring, a technique borrowed from cognitive behavior therapy and rational emotive training.

b) Dialogues or pedagogy in which unhelpful cognitions are exposed and replaced.

c) Forums in which learners' predictions about math can be tested against results, (Computer environments or lab materials).

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REPRESENTATION SYSTEMS AND MATHEMATICS

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RESUME

Cette étude fait partie d'une série touchant la représentation et l'enseignement des mathématiques. Elle propose le développement d'une théorie globale et cohérente de la représentation et des systèmes de symboles en fonction des deux principes suivants: 1) la nécessité d'aborder immédiatement les difficultés, aussi persistantes que bien connues, que rencontrent les étudiants à opérer une traduction entre les diverses représentations des idées mathématiques, d'une part, et entre les représentations mathématiques et non-mathématiques, d'autre part; 2) la reconnaissance que les mathématiques, aussi bien que l'activité cognitive nécessaire à leur pratique et à leur emploi, prennent racine dans la représentation. Nous présentons une stratégie pour l'élaboration de la théorie, tout en traçant l'esquisse des caractères déterminant le concept de la "représentation".

Nous abordons cette théorie par l'explication des liens profonds qui existent entre l'idée de la représentation et le contenu des mathématiques elles-mêmes.

INTRODUCTION

This paper is a preliminary argument for an alternative theoretical perspective on certain key issues in mathematics education, a perspective that in the most general terms might be characterized as providing a linguistic/semiotic complement to the exclusively cognitive approach now current. The paper is part of a series dealing with the idea of representation in mathematics education that is motivated by a tandem of practical concerns and theoretical needs. The practical concerns center on the well known deep and continuing difficulties experienced by students in translating between alternative representations of mathematical ideas and between common experience and mathematics. These concerns are magnified by the radical new range of representational opportunities offered by computers and the heightened requirement for systematic knowledge regarding the ways that such representations function and interact. The theoretical needs are a consequence of these practical matters coupled with the fact that this author knows of no comprehensive, systematic theoretical framework of symbol systems and representation systems capable of supporting the kinds of understandings necessary to solve the problems just alluded to.

Some mathematics education researchers have, in response to this need based on work in a particular area, developed local theories, e.g., to account for the differential ability of certain representations of rational numbers to carry certain aspects of the arithmetic of rational numbers better than other representations, (Lesh, et al, 1983; Chaffe-Stengel & Noddings, 1982). There have also been sporadic efforts to address the issues more broadly, the most successful example of which is probably the collection edited by Skemp (1982). However, a coherent and unifying theoretical context is lacking. This factor may be of critical importance to the efficacy of empirical work and its ultimate impact on the curriculum and instruction (Kilpatrick, 1981). Others have devoted considerable effort to the study of symbol systems outside of mathematics, (Langer, 1942; Cassirer, 1950; Bruner, 1973; Goodson, 1968; Salomon, 1979; Gardner, Howard & Perkins, 1974; Gardner, 1978) among others. Although some of these works are classics and monumental in their scope, they have little to say directly about the issues that would inform the mathematics curriculum.

The overarching assumption of the line of inquiry advocated in this paper is that more direct and systematic attention needs to be paid to the ways we use symbols and combinations of symbols in mathematical

representation systems and the ways these systems relate to one another. An adequate theory would define general properties of symbol and representational systems themselves (as opposed to their potentially variable informational content). It would provide systematic criteria and vocabulary for discussing how the different representation systems vary in their ability to encode certain mathematical ideas. These would be characterized in terms of psychologically identifiable structures and processes, such as the amount or kind of processing required to extract certain information from a particular encoding or to translate a given piece of information from one system to another. It would provide a developmental trajectory for representational skills that would interface with descriptions of the other symbolic competences synchronically developing within the student, especially natural language competences (Bardner, 1978; Nesher & Schwarz, 1982). And it would necessarily connect with the existing corpus of knowledge in mathematics education.

To construct and support such a theory will not be an easy task and will surely take a number of years. Below we outline a strategy for beginning the effort. But first we would like to be a bit more specific regarding the types of questions that this kind of theory might illuminate.

REPRESENTATION IS UNDERREPRESENTED

There is a common tendency to underestimate the role of representation systems in the standard curriculum. For example, we usually assume the mathematics curriculum in the first 8 years of school is about numbers, whereas the actual school work is about a particular representation system for numbers - the base ten placeholder system - and its properties. The essence and power of numerical algorithms reside in the freedom to deal only with the representations of numbers without regard to the numbers they represent. Because of the construction of the representation system and the design of the algorithms, we can then be confident that the symbol produced by the algorithm represents the actual answer. Some perennial difficulties get seated in a different context when seen in these terms. For example, the failure of students to estimate or maintain an order of magnitude sense of a calculation (Davis & McKnight, 1980) can in fact be seen as a failure to cross between symbol and referent. There are aspects of the skill automaticity or "rote vs meaningful learning" question that can be stated in these terms as well (Nachreuth, 1982, 1983).

The curriculum likewise ignores what may in some

instances be an important distinction between those properties of numbers that are sensitive to the representation system vs. those properties that are relatively independent of the representation system, e.g., primeness of numbers. We give virtually no explicit attention at any level in mathematics education to the relation between the transparency of certain mathematical properties or operations and the representation in which they are encoded, e.g., the Cantor "excluded middle" set is very transparently constructed using the base three representation of real numbers and some of its properties likewise are easily deduced from this representation, but not from others. Sensitivity to choice of representation is well known in problem solving. (Recall the classic question regarding the monk who climbs to a mountaintop in one 12 hour span, prays 12 hours and then spends another 12 hours descending on the same path he used for the ascent. Is there a place on the mountain where he is at exactly the same place descending at exactly the same time of day as when he was ascending 24 hours earlier?) Similarly, we omit from explicit instruction why some representations carry certain quantitative relationships more efficiently than others, as with rectangular vs polar coordinates. Likewise, we seldom deal with the critical issue of scaling when graphing, especially the relation between non-linear scales and the simplicity of the graphical representation of quantitative relationships, e.g., logarithmic scales and exponential growth/decay.

Naturally, there is a tight connection between the omissions of the curriculum and the omissions of the research community. However, as mentioned above the increasing availability of computer representations will magnify the importance of these omissions because instructors will have an enormous new variety of representations of mathematical ideas to work with, some of which may support one another, others of which may interfere with one another. The students of the near future likewise will be choosing how to represent given relationships. This skill in choosing or building representations, together with interpretive skills, will likely outstrip computational skills in importance by a wide margin. Both teachers and students will need more than uninformed intuition to guide them. They will need systematic, reliable and psychologically valid knowledge about representations.

Another relevant problem area is translation between non-mathematical representations and mathematical ones. As noted in (Kaput & Saxe-Knight, 1983) and elsewhere, student difficulty in such translation tasks is legend, yet very little focal attention is given to the teaching of how the various representation systems work and how

they relate to one another. On the other hand research indicates that this is a critical component in the ability to translate between systems (Sies-Knight & Kaput, 1983). A good theory of representation and symbol systems would be able to characterize in systematic ways the failure of systems to fit together smoothly. This is reflected in the "Students-Professors Problem" reversal error, where certain (multiplicative) quantitative relationships are expressed in mutually opposing ways in natural language vs algebra (Clement, 1982; Kaput & Sies-Knight, 1983). Similarly, certain representations are "cognitively orthogonal" when given pictorially vs. graphically, as with the picture of a cyclist traveling over a hill vs the graph of her speed as a function of time (Sies-Knight & Kaput, 1983). We need more coherent ways of thinking about such phenomena.

STRATEGY FOR A THEORY

We now turn to the strategy for constructing the theoretical framework advocated earlier. The fundamental premise is that the root phenomena of mathematics learning and application are concerned with representation and symbolization because these are at the heart of the content of mathematics and are simultaneously at the heart of the cognition associated with mathematical activity.

Our working premise is that the hardest part of the theory development will be establishing the psychological/linguistic reality of the constructs associated with external symbol creation and its bonding with internal cognitive structures and processes. The opaqueness of this bonding suggests that it be treated temporarily as a black box until enough conceptual and methodological tools are in hand to open it. Instead we begin informally, first with an examination of the content of mathematics itself from the representational perspective without regard to psychological, linguistic or curricular constructs. We then take a first step in the direction of a psychological and linguistic theory by devising a quasiformal notion of representational system that makes sense in terms of the existing core mathematics curriculum. We will show how basic mathematical ideas relate when viewed from this perspective and discuss the various characteristics of these systems, especially with regard to their symbol systems, which play a critical functional role. The third stage in the evolution of the theory is an extension of the quasiformal notion of representation system to include some basic psychological phenomena connected with the representation of root mathematical concepts such as number, and the role of the ambient

representation systems of natural language and visual images. Subsequent steps will examine how symbol systems build on one another and how intersubjectively shared external symbols interact actively with their internal referents.

DEFINING "REPRESENTATION"

There is real danger in stretching an undefined term such as "representation" across domains as different as formal mathematics, cognition and epistemology. We deal with this danger by defining the term more precisely in several stages of increasing specificity and formality. We begin with a proto-definition that serves as a definitional framework into which more specific definitions can be fit later.

We follow Palser (1977) and, in fact many others, in assuming that any concept of representation must involve two related but functionally separate entities. We call one entity the representing world and the other the represented world. There should then be a correspondence between some aspects of the represented world and some aspects of the representing world. Hence any particular specification of a representation should describe the following five entities:

- (1) the represented world,
- (2) the representing world,
- (3) what aspects of the represented world are being represented,
- (4) what aspects of the representing world are doing the representing, and
- (5) the correspondence between the two worlds.

In many of the interesting cases one or both of the worlds may be hypothetical entities or even abstractions. There are four broad and interacting types of representation that can be accommodated under the above general framework: (1) Cognitive and perceptual representation, (2) Explanatory representation involving models, (3) Representation within mathematics, (4) External symbolic representation.

Our strategy for theory development can be viewed in part as aiming for integration of all 4 types of representation. In (Kaput, 1983) we introduced the notion of representation and dealt with (2) in some detail, focussing on computer models of mental representations and the question of explanatory adequacy. Space limits us to a sketch of how the content of mathematics can be viewed as formal representation (3). The next step in our inquiry, a description of the core school mathematics curriculum in

teras of representation systems, appears in another paper (in preparation).

REPRESENTATION IN MATHEMATICS

Mathematics may be regarded as the science of significant structure. Thus mathematics studies the representation of one structure by another, and much of the actual work of mathematics is to determine exactly what structure is preserved in that representation. This representation is formally independent of the form of the symbols used because the structure itself is treated as an abstraction or idealization. Of course, the symbols do play a crucial role at every level of mathematical representation as suggested earlier. We now outline several of the more common kinds of representation in mathematics.

1. The most frequently used construction in all mathematics involves structure-preserving mappings from one mathematical structure to another of the same general type. For example, a homomorphism $f: A \rightarrow B$ between groups is a representation in the sense defined earlier, where the mapping is the correspondence and, depending on the particular application at hand, either A or B can be the represented world. The aspects of each involved in the representation are the respective group structures as embodied in the operations and the rules they obey. These can be entirely abstract, or in other cases relatively concrete in the sense of being a group whose elements and operation are a familiar construct. For example, B may be a group of matrices with the operation of matrix addition. There are whole branches of mathematics based on the development and application of such representations, e.g., the representation theory of finite groups. An example of such is given by Cayley's Theorem, which states that any group is isomorphic to a group of permutations on the set of its elements. This example illustrates a stronger form of representation wherein the correspondence is an embedding.

2. Most of the standard constructions of abstract algebra can be regarded as providing a simplified representation of a given abstract object, as with the quotient of a group by a (normal) subgroup, which provides a simplified version of the original group with all the complexity of the subgroup "factored out." A similar remark applies to the role of Fundamental Homomorphism Theorems for algebraic objects of most any sort.

3. Frequently an object A is represented by B via an isomorphism $f: A \rightarrow B$ where B is constructed from simpler structures of the given type. For example The Fundamental Theorem of Finite Abelian Groups asserts an isomorphism between any finite commutative group and a

direct product of cyclic groups of integers.

4. Some representations are defined in a more "internal" fashion, as in the Fundamental Theorem of Arithmetic (any integer has a factorization into a product of primes in an essentially unique way.) Here the representation is not defined by a structure-preserving mapping but by an equation. As with the examples in (3) the Fundamental Theorem of Algebra (every polynomial has a factorization into linear factors in an essentially unique way), and a whole host of similar types of constructions, a given object is represented by simpler, more atomistic objects.

5. A class of representations occurring all across mathematics involves approximation of a given object by another simpler or more primitive object, as with the Taylor Polynomial or series approximations of suitably differentiable functions by polynomial functions or series. The correspondence is the approximation itself. Much of classical analysis revolves around such representations.

6. Another class of representations disambiguates a particular aspect of a mathematical object from the other features of the object by creating a new object of a different kind which represents that aspect cleanly. For example, the Fundamental Group of a topological space, describes the "holes" (simply-connectedness) of the space in a precise way. Other examples of such (functorial) representation assignments abound.

7. Mathematics also uses one mathematical idea to represent another, as with the Fundamental Theorems of Calculus, where accumulation is represented by rate of change or vice-versa. The two most commonly given versions differ only in which is taken as the representing world and which is the represented world.

8. A slightly different kind of representation is used in classic logic, where, given an axiom system with a collection of undefined primitives, one makes a formal interpretation of the primitives in some other, presumably more familiar system S which serves as a representation ("model") of the axiom system. In this way axioms are tested for consistency, categoricity, dependence, etc.

The frequent use of the phrase "Fundamental Theorem" in the above discussion is not an accident. Most of the results that mathematicians regard as truly fundamental are easily classifiable as representational. Also, such actual research in mathematics is an attempt to extend such representations to new mathematical domains involving more esoteric objects. The representational goals remain the same.

It should be apparent that the idea of representation is continuous with mathematics itself.

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AXIOMS AND INTUITION IN MATHEMATICAL KNOWLEDGE BUILDING

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RÉSUMÉ

En 1983, Hart établit une relation entre la performance dans des tâches mathématiques accomplies par des étudiants d'écoles Secondaires et leur possibilité de recourir à des méthodes simples ou enfantines. Cette communication est une exploration théorique de la nature des sources de ces intuitions en mathématiques. Le concept d'intuition est décrit comme étant la somme d'imageries, de mécanismes constructifs et de l'emploi d'un langage informel. La thèse majeure de cette communication est que les systèmes d'axiomes en mathématiques sont indicatifs de problèmes relatifs au mécanismes d'acquisition, soulignant l'importance de la pensée intuitive. Cette hypothèse est appliquée au dénombrement et utilise Peano pour faire ressortir l'importance de la coordination entre l'unité et le succes-
seur dans l'apprentissage du calcul. L'étude du comportement des enfants à l'égard des opérations arithmétiques a permis de relier les qualités intuitives du développement des nombres rationnels aux axiomes se rapportant au quotient.

Enfin, le problème plus général de l'élaboration d'un programme à partir de cette perspective est discuté brièvement.

In discussing the performance of British secondary school children on a paper and a pencil test of rational number knowledge and thinking, Hart (1983) noted that there were identifiable "hard" and "easy" items. The easiness of the item appeared to depend "on the availability of naive or child methods which [in the case of rational numbers] depend on whole numbers, halves, counting and addition." One might say that these naive or child methods are a form of mathematical intuition.

Several questions arise out of the observation above. What is the nature of mathematical intuition? Is it useful in building personal mathematical knowledge and if so, how? If it is useful and if one wishes to study such intuition and/or to build such intuitive methods or thinking, are there guides which point to intuitive behaviours? The first of these questions has received historical attention under the rubric of insight. There is contemporary work in this area as well (eg., Fischbein (1983)). The first section below contains some remarks on this issue as well. In answering the second question, one might think that formal instruction in mathematics might by-pass or actively try to suppress naive methods in favour of a more orderly development of mathematical structures. It is an assumption of this paper that a person's mathematical knowledge must be about experiences both externally referenced and internal and that formal mathematical knowledge must be based on informal knowledges for the person. Fischbein (1983) suggests that counting is an intuitive basis for whole number knowledge in that it helps the person build representations of number which are sympathetic to the natural functioning of the person. Although counting and to a certain extent, ideas of infinity and some geometric ideas have been studied from the point of view of intuition, this is not true for mathematical concepts or systems. It is the purpose of the second part of this paper to explore the relationship between axioms and personal knowledge building and hence mathematical intuitions.

THREE SIDES OF INTUITION

In other work I have developed a five faced model of mathematical knowledge building (Kieren, 1981). A psychological face addresses itself to developmental and information processing issues. A mathematical face is based on the premise that useful personal mathematical knowledge comes from experience with the variety of mathematical variants of the concepts set as well as with the contexts in which the concepts might manifest itself. The notion of mathematical intuition used in this paper is based on the remaining three faces. The third face pertains to constructive mechanisms. These are specific developable thinking tools such as counting, dividing up equally, forming ratios and functional relationships, equidecomposing and generating procedures. The fourth face involves having, experiencing and creating images of mathematical ideas. The fifth face involves the ability to represent mathematical ideas with formal, and especially for this discussion, informal symbols. These five faces are not independent and found mathematical knowledge for a person, rather than represent it completely. While not ignoring the psychological or mathematical face implications for intuition, mathematical intuition is taken here to mean the coordination of constructive mechanisms, imagery and appropriate (usually informal) symbol usage. This informal usage is related to what Frye (1981) refers to as hieroglyphic (object tied) and metonymic (analogical) uses as opposed to analytic usage. For example, a child who has a limited knowledge of partitioning, an image of adding as putting together quantities and an analogical use of simple fraction language might do the following.

"three quarters plus one half, umm, a half is a quarter and a quarter; one quarter takes you up to one so one and one quarter."

Although it is beyond the scope of this paper to do so, this three sided notion of intuition can be applied in mathematical situations at a more advanced level and using language in a more sophisticated if still analogical manner. Further, though this example represents

behaviour which is not deliberately schooled, such coordination might be the object of instruction. Friedman (1978), for example, suggests that intuition is indeed a possible product of a deliberate educational enterprise.

THE ROLE OF AXIOMS

If we are to build the mechanism, image, informal symbol use triangle into a curriculum for a mathematical topic, what mechanisms, images and symbols might we choose? A formal description of a mathematical topic includes axioms and theorems which are proveable from the former. Philosophically, axiom sets are studied for their consistency (they have a model) and their independence. Thus axioms represent a minimal set of assumptions for a system. In the older sense, axioms represented some "obvious truths." This older sense may be more pertinent to personal knowledge building. It is a major thesis of this paper that

axiom sets point to significant knowledge building problems.

It is corollary of this theorem that axiom systems are useful in pointing to the characteristics of mathematical intuition for a given topic. The potential of this axiom and corollary are explored in two examples below.

EXAMPLE ONE: COUNTING NUMBERS

The classic axioms for the counting numbers are those of Peano. These axioms describe and determine relationships among three undefined terms number, unit and successor. It is rather circular to note that in terms of constructive mechanisms, the Peano Axioms point to counting. Still it is interesting to note that the number knowledge building skills are related to the axioms in a specific way. For example, a child has to overcome the problem of knowing where to stop when counting a set of objects which are not ordered. This relates to the axiom that every element has a successor. In terms of imagery the axioms indicate linear or stepwise images. Two forms of informal language use are implied by the system especially

for young children. The first arises from the undefined term "number"; at an early age a child uses number words heiroglyphically coming to associate at least some number names with object sets. Counting also represents an analogical use of language with the sequence put for a linear rhythmic pattern as well as elements of this sequence being put for objects. At first this association is object directed. Later a counting sequence is used as an expression of thought put for but independent from the object sets.

The axiom system also points to the unique and necessary contributions of successor and unit in intuitive number thinking. Thus the verbal act of counting (even with an idiosyncratic order of number words) is an intuitive reflection of the axiom that every element has a successor. The primitive act of pointing while counting both acts to identify object units and to set up the correspondence between verbal units (occurring sequentially) and objective units. The behavioural manifestations of successor noted here and their consequences have been discussed in the work of Gelman and Gallistel while those with respect to unit have been studied in detail by Steffe and his associates. The applications of axiomatic analysis to intuitive number thinking indicates the complimentary nature of these concepts and the associated research.

EXAMPLE TWO: RATIONAL NUMBERS

The axiom system which governs the rational numbers is that of a quotient field. Two principle concepts used in the system are rational number (solutions to $bx = a$, a, b integers, $b \neq 0$) and equivalence. The axioms also show addition and multiplication to be independent concepts. Two other key ideas are inverse (used in two ways) and unit (serving both as the basis for additive quantities and as the multiplicative inverse). The quotient nature of rational numbers points to partitioning (dividing up equally) as a key constructive mechanism in rational number knowledge building. This mechanism should act additively (a partition seen as a sum of results of partitions; partitioning underlying addition in some way).

It should act multiplicatively (a partition can be a product of partitions). Preliminary studies have found evidences of partitioning being used by subjects ages 5-14 in both these ways (Pothier (1981), Kieren and Nelson (1981), Kieren (1983)). Two image types are suggested by the axioms - an additive image (quantities being put together) and a multiplicative image (a size transformation or a many-many correspondence). It is useful to note that the static part whole images used in many texts do not match the axioms and probably do not aid rational number thought development as they might. Such part whole images are also associated with one type of hieroglyphic language use. An early analogical use of fraction language is the word 'half' being put for the act of dividing (usually in two parts). A later intuitive analogical use of language reminiscent of the Egyptian use is as follows:

" $\frac{1}{4} \div 3$ is one plus a quarter plus a third".

The axioms suggest the need for informal (but systematic) fraction language where language is put for additive objects (as above) and also where language elements are put for multiplicative acts.

The analysis above has focussed in the main on partitioning and its relationship to the multiplicative and additive nature of rational numbers. Space does not permit the discussion of mechanisms, images and language associated with equivalence, inverse, identity, etc. Such intuitions can be discussed in terms of the role of the rational number sub-system $[(1/2)^k, k \text{ an integer}]$ and the unit fractions.

The key idea of this paper is that there is a close relationship between the axioms of a mathematical system and the elements of intuition of the system expressed in terms of constructive mechanisms, images and hieroglyphic and analogical language use. This thesis still has the status of an unconfirmed speculation. The two examples are an attempt to show the potential of this thesis. The thesis also appears congruent with the van Hiele levels which suggest axiom related whole object play and hieroglyphic language use followed by property and relationship development accompanied by analogical

language use as precursors to formal axiomatic work and demotic language use. Considerable research with the two systems exemplified above as well as with systems from algebra (eg., vector spaces) and analysis are necessary to first establish the credibility of the thesis and then its validity.

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CONSTRUCTIVIST APPROACHES TO TEACHING MATHEMATICS
AND SCIENCE AT THE COLLEGE LEVEL

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RESUME

L'université de Massachusetts/Amherst a créé un centre pour étudier les approches cognitives de la didactique de la mathématique. Les cours de ce centre s'insèrent tous dans la perspective épistémologique de la philosophie constructiviste. La méthode utilisée pour enseigner les notions fondamentales constitue l'aspect le plus remarquable de ces cours. Au lieu d'éliminer la confusion résultant de l'adoption de perspectives différentes au cours de la construction d'une notion, nous tâchons plutôt de l'encourager et de nous en servir. Nous croyons que cette étape de confusion est essentielle pour le développement de la compréhension.

PHILOSOPHY

The University of Massachusetts at Amherst has recently established a Center for the Study of Cognitive Approaches to Mathematics Instruction. The goal of this center is to find ways for teaching mathematical reasoning to students who have previously had difficulty with the subject. Currently two courses are involved, Math 010 and Math 011. They are based on a philosophy of education which is substantially different than that commonly practiced in mathematics. Fundamentally it assumes a constructivist view of learning. That is, it assumes that meaningful learning can only take place when the learner actively constructs concepts and relations. Expository forms of teaching are discouraged, particularly when working with new material. It is this last point that is hardest for non-constructivists to accept since they feel it is important to tell the learner the fundamental rules and concepts to insure there is no confusion over these basics. As constructivists we believe that such confusion is unavoidable and that it is only after the learner has thought through many alternative basic principles that she or he is ready to learn from more traditional formats. The teacher's leverage, particularly in the initial phase, is very limited. It is possible to present students with problems to think about, but not to control how or whether they think. However, the better the teacher understands the student's current conceptual system the more likely the teacher is of being able to select problems which will stimulate the student's curiosity and also focus thinking in desirable directions. Discussions are very useful at this phase as the communication process forces people to clarify their ideas while at the same time broadening their view through exposure to other people's perspectives. A highly skilled teacher can take part in such discussions by acting the role of another student. However, it is often easier and more beneficial to encourage discussion between peers. Nonconstructivists worry that this is nothing more than the blind leading the blind. That may be true but one needs to remember that the objective is to learn the terrain not to cross it in the shortest possible time. Nevertheless, this is a recurrent

issue for the constructivist teacher, few students understand constructivism or see the dangers of rapid content coverage.

Students often are annoyed by the need to struggle on their own and only a skilled teacher can help them see the point of such endeavors.

SYLLABUS

Math 010 is designed for students who have found mathematics very difficult and who need a complete review. It assumes no prerequisites. Texts for the course are Problem Solving and Comprehension and Developing Mathematical Skills both by A. Whimbey and J. Lochhead. The latter covers arithmetic of whole numbers, fractions, decimals, percents, and negative numbers; exponents; simple algebra, and elementary geometry. A unique feature of the text is its emphasis (and chapter) on translating between English and mathematical notation as a separate step, independent from solution generation. A wide body of research, e.g., Clement, Lochhead and Monk (1980), has shown that coverage of this important translation skill is utterly inadequate in most existing mathematics instruction. In Math 010 we view deficiencies in student mathematical knowledge to be symptomatic of inadequate strategies for the learning and conceptualization of mathematical ideas. We therefore place primary emphasis on the development of these strategies. The course begins with the pair problem solving strategy used in Problem Solving and Comprehension (see also Lochhead, 1983). Through this approach students learn to communicate with each other about every aspect of their mathematical reasoning: how they conceptualize numbers (e.g., fractions), operations and problems. These conversations make students aware of their own reasoning and learning processes as well as those of others. While there are many functions for such reflective thinking one of the most important is the role it plays in the development of an appropriate self confidence in one's own work. Students tend to view mathematical correctness as something which can only be determined by external authority. As long as they persist in this view meaningful learning

is impossible.

The teacher's role during this part of the course (which for some students extends throughout the entire semester) is to encourage student discussion, to demand greater precision of communication and to stimulate deeper analysis of problems and concepts. The teacher must avoid the role of mathematical authority and confirms or refutes student problem solutions only after the students have thoroughly evaluated them and reached a sense of commitment. For the teacher to do this well requires several special skills, one of which is the ability to understand how the students are thinking in sufficient detail to respond with questions or suggestions that are meaningful and exciting to the students.

Throughout Math Q10 the class structure is primarily based on students working together in pairs according to the rules of pair problem solving, or cooperatively in small groups. The teacher's function is rarely expository. Most instructional activity is in the form of coaching and of analyzing areas of confusion which are then addressed not by explanation but rather by the construction of appropriate problems. For example, failure to grasp the commutability of fraction multiplication might be addressed through problems asking for figural representations for expressions such as $1/3 \times 2/3$ and $2/3$ of $1/3$.

Although Math Q10 covers material much of which should have been learned in elementary school, we do not "spoon-feed" or "baby" students. We feel it is important for us to show the same respect for our students' intelligence that we wish them to develop in themselves. This is done in several different ways. Homework in Developing Mathematical Skills is assigned but not collected; students must grade it themselves from posted answer sheets. Thus learning and drill of basic math facts is something students must do on their own or with classmates. We have found that expecting this level of responsibility has produced better results than had earlier more authoritarian approaches. It is an approach which has

also proved effective in the Guided Design Curricula (Wales & Stager, 1977); but it does require skill on the teacher's part to insure that less motivated students are pressured to keep up.

Two types of homework are collected and graded. One consists of difficult word problems which require students to go substantially beyond the level of comprehension demanded by the text. The second is a set of writing assignments, called thought process protocols, in which students must describe in detail the reasoning processes they went through (including blind alleys) while working a moderately difficult word problem. Correct solutions are not required for protocol problems; grades are based solely on the quality of the protocol as a detailed trace of a thought process. Several example exercises are given at the end of this paper. While most of our students are able to solve these problems working cooperatively in groups they frequently are unable to get help from advanced students, including math majors, and sometimes faculty. Their discovery of this phenomenon, while initially distressing, eventually leads many of them to an upward re-evaluation of their mathematical potential.

MATH 011

Math 011 is the second semester continuation of Math 010. It assumes students have mastered the problem solving and learning skills taught in that course and uses these to develop fluency with algebra and graph interpretation. The text is Algebra I by Paul Foerster. In Math 011 we notice substantial improvement in the mathematical maturity of the students. The course has a similar structure to Math 010 with most class time devoted to group work on textbook problems or harder extensions of them. By now students are so used to working together that they often do not need, desire, and in some cases tolerate, teacher assistance. Usually they have full confidence in their own solutions, but they now become interested in discovering many alternate solution paths and in comparing different methods. The teacher's role is in part

similar to that of the Math 010 teacher but differs by requiring a greater mathematical sophistication and more agile reasoning in order to follow student thinking. We also make greater use of a lecture-discussion format for presenting new topics. Because of the training students received in Math 010 on question asking and in demanding solid conceptual understanding, the teacher's role in such discussions can be rather stimulating.

EFFECTS

Since the objectives of courses such as Math 010 and 011 are not entirely consistent with those of the more standard variety, comparative evaluation is extremely difficult. A four year study of a course for engineers with similar philosophical underpinnings, but very different content, revealed no significant difference in grade point average. However, over 50% of the students in the constructivist course remained in the Engineering program versus less than 35% of the control group. A recent comparison of Math 010 and 011 students with a traditional course revealed no significant differences in performance on standard Algebra content (though non significant differences always favored the constructivist groups). Unfortunately, no comparison was made on "confusing" word problems because the teacher in the traditional class found it necessary to rewrite those problems to remove the "ambiguities".

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SAMPLE PROBLEMS FROM MATH 010 AND 011

1. In a certain population two-thirds of all men are married but only $\frac{3}{5}$ of all the women are married. What fraction of the population is single? Are there more men or women?
2. In a certain city four-fifths of all homeowners are men and $\frac{2}{3}$ of all men who own homes are married. A full $\frac{3}{5}$ of all men are property-less even though half of them are married. What fraction of the men are single homeowners?
3. Write the following two sentences in mathematical language.
Each week Arnold spends twice as much time watching TV as he does listening to his stereo. He listens to his stereo for 12 more hours than he spends studying.
Now write an equation that relates the amount of time Arnold watches TV to the amount of time he spends studying.
Solve the above equation to find how much time Arnold spends studying each week if he spends 6 hours each day watching TV.
4. In a certain dynasty one-third of the kings were of the same name, one-fourth of another, one-eighth of another, one-twelfth of a fourth, and there were five besides. How many kings were there of each name?
5. A certain number of persons were divided into three classes, such that the majority of the first and second together over the third was 10 less than four times the majority of the second and third together over the first; but if the first had 30 more, and the second and third together 29 less, the first would have outnumbered the last two by one. Find the number in each class when the whole number was 34 more than eight times the majority of the third over the second.

(Problems 4 & 5 are from Algebra by Todhunter, Macmillan, 1881 and are among a set of such problems being republished in Beyond Problem Solving and Comprehension, Whimbey and Lochhead, Franklin Institute Press, 1983.)

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MATHEMATIQUE ET AFFECTIVITE

(Recherche sur divers modes de relation à l'objet mathématique)

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ABSTRACT

In the present paper, the author studies different types of relations used by the pupils, the mathematics teachers and mathematicians concerning the mathematical object. He shows how this object is dressed with phantasms, apprehended through varied defense mechanisms, and finally used in each individual's psychical dynamics according to the structure of his personality.

This research is mainly conducted using two methodologies, one clinical, the other experimental.

- About a hundred taped clinical interviews (sixty interviews with students aged 16 to 18 and forty with teachers and mathematicians) have led to clinical analyses.

- A research using questionnaires has been conducted in order to support and to diversify the results of the preceding analyses. These questionnaires have been used with:

- 2000 students aged 16 to 18 from 5 countries (France, Belgium, Canada (Québec, Ontario), the United States, and Great Britain)
- 1000 French mathematics teachers;
- 110 Literature teachers (control group).

These different types of questionnaires have been studied using principal component and hierarchical analyses.

RÉSULTATS :

RELATION D'OBJET "... mode de relation du sujet avec son monde, relation qui est le résultat complexe et total

- d'une certaine organisation de la personnalité,
- d'une appréhension plus ou moins fantastique des objets,
- et de tels types privilégiés de défense."

Laplanche et Pontalis p. 404

Dans cette recherche l'auteur étudie en premier lieu divers fantasmes recouvrant l'objet mathématique soit à un niveau collectif à travers les expressions utilisées dans les mass-médias, soit à un niveau individuel dans le discours des élèves ou des professeurs soit encore dans le choix des signifiants utilisés dans le discours mathématique lui-même. Il montre, en particulier, comment l'objet mathématique peut servir d'objet à diverses pulsions comment il peut être le substitut d'objet partiel ou d'objet total, comment il peut être utilisé dans l'expression de fantasmes oedipiens ou de fantasmes originaires. Il termine cette partie en proposant des hypothèses sur les mécanismes de "fantasmatisations" des mathématiques : déplacements, chaînes associatives et "glissement" de signifié sur un même signifiant.

Dans une deuxième partie l'auteur étudie différents mécanismes de défense utilisés par les élèves à l'égard de l'objet mathématique. Il met en évidence deux groupes de mécanismes : l'un phobique, l'autre maniaque. Il montre comment ces groupes se retrouvent dans chaque pays avec parfois certaines prédominan-

ces. Il montre comment l'anxiété à l'égard des mathématiques est un facteur indépendant de l'attitude de l'élève mais résulte du choix des mécanismes de défense utilisés par ce dernier.

Dans une troisième partie l'auteur confronte les champs de représentation des mathématiques chez les professeurs de mathématiques avec leur attitude à l'égard de leurs élèves. Il apporte un ensemble d'arguments tendant à montrer la corrélation existant entre la représentation que se fait le professeur de l'objet mathématique (par exemple comme loi structurante) et ses choix pédagogiques (importance de la rigueur du langage, de la cohérence du cours, dans ce dernier cas). Il propose ainsi diverses utilisations de l'objet mathématique dans la dynamique psychique de l'individu, utilisation qui, en définitive, conditionne à la fois la représentation qu'il a de l'objet mathématique et ses choix pédagogiques.

Une analyse factorielle et une analyse hiérarchique permettent ainsi de mettre en évidence quatre modes de relation à l'objet mathématique : un mode persécuteur, un mode schizoïde, un mode anaclitique et un mode de maîtrise.

Dans une dernière partie l'auteur étudie l'heuristique mathématique et le travail de l'inconscient. Il y propose de retrouver à travers des entretiens cliniques faits avec des mathématiciens, les cinq phases de l'oeuvre créatrice analysées par le professeur Anzieu.

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LEARNING MATHEMATICS AND NEGATIVE INSTANCES

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RESUME

Le raisonnement à partir de "contre-exemples" est considéré comme un élément important de la maturité mathématique. Peu de livres ou de textes illustrent les contre-exemples. La recherche en psychologie appuie le recours à des cas négatifs sauf pour les concepts de conjonction (conjunctive concepts). La recherche en psychologie éducative appuie le recours à des cas négatifs même pour les concepts de conjonction. La recherche en éducation mathématique présente des résultats mixtes. Le caractère de fréquence des dimensions non pertinentes a été retenu comme variable susceptible d'expliquer la différence dans les résultats entre tâches psychologiques et tâches mathématiques. L'attention accordée au contexte, les caractères pertinents ou non pertinents, les connecteurs logiques et les cas négatifs sont susceptibles d'être importants dans la plupart des apprentissages mathématiques qui ne relèvent pas des habiletés. Les problèmes de programmation d'ordinateur en mathématiques apparaissent les plus appropriés pour amener chez les élèves des stratégies d'apprentissage de concepts adéquates et appuyées sur la recherche. Notre projet est d'étudier systématiquement l'utilisation de la programmation d'ordinateurs pour enseigner les concepts fondamentaux des mathématiques.

In order to identify fundamental issues in learning mathematics, Shumway (1968) interviewed a number of mathematicians in an attempt to define "mathematical maturity." Among the fifty or so attributes identified was the commonly cited "ability to reason from counterexamples." Two books by mathematicians (Gelbaum & Olmstead, 1964; Steen & Seebach, 1971) lend support to a role for counterexamples in the learning mathematics. While many mathematics teachers may regularly use counterexamples in their teaching, an examination of published textbooks suggests, on the whole, printing counterexamples in textbook materials is seen as inappropriate. Teachers often suggest reluctance to use counterexamples derives from a behaviorist view of learning suggesting examples which do not work would be unhelpful. It is possible a focus on skill learning, and a minor interest in concept learning and problem solving would also lead one to deemphasize counterexamples.

The research in psychology beginning with Smoke (1933), culminating with Bourne and Guy (1968), and continuing with Kellogg (1981) has carefully documented the effects for negative instances in learning conceptual rules and the role feature frequency may play. Summaries of the research (Clark, 1971; Bourne & Dominowski, 1972; and Erickson & Jones, 1978) consistently report treatments of both positive and negative instances are favored for all conceptual rules except affirmation and conjunction (i.e., disjunctions, conditionals and biconditionals). There appears to be little work on the role of negative instances in psychological problem solving research, although Shumway (1982) makes the case that the psychological concept learning research is most relevant to mathematical problem solving research.

The work of educational psychologists is most supportive of the use of negative instances as part of an instructional strategy. (Gagne, 1970; Markle, 1975; Tennyson & Park, 1980). One of the strongest supports for the role of negative instances by educational psychologists is in the work of Tennyson, Woolley, and Merrill (1972). However, none of these studies were conducted with mathematical tasks.

Sowder (1980), in summarizing and reacting to the mathematics education research on concept and principle learning regarding the use of nonexamples, suggests there are some puzzling issues yet to be resolved although there seems to be support for the use of nonexamples in instructional strategies. Sowder identifies a re-occurring theme that subjects seem to have some inability to use information from nonexamples.

Shumway and colleagues (Shumway, 1971, 1974, 1977; Shumway and Lester, 1974; White, Wilson, & Shumway, 1981; Stout, 1982; Wilson, 1982; and Shumway, White, Wilson, & Brombacher, 1983) have been working for some 12 years on what they describe as the discrepancy in effects for negative instances. Apparently negative instances are favored on the conjunctive mathematics tasks but not the conjunctive psychological tasks. Their most recent studies (Stout, 1982; Wilson, 1982; and Shumway, et al., 1983) suggest feature frequencies as the explaining variable for the differences. By manipulating the feature frequency of irrelevant attributes both results regarding negative instances (favoring and not favoring) were produced in both contexts, the psychological tasks and the mathematical tasks. The variable manipulated was the feature frequency in the irrelevant dimensions of a concept. If there exist dimensions whose features do not vary often, negative instances become most valuable in eliminating irrelevant dimensions. It is our view any learning research which is not strictly skill learning research, must attend to the following considerations:

1. Identify concepts.
2. Specify universe over which concept is defined.
3. Identify relevant and irrelevant features.
4. Identify logical connectives needed to define concept.
5. Include negative instances as part of any learning activities.
6. Test knowledge of concepts with new instances (both positive and negative) sampled from the universe.

The only circumstances known to us where negative instances may be inappropriate are for conjunctive concepts where all features of each

irrelevant dimension are equally likely. We find many examples of such concepts in psychological research, but none in the mathematics classroom.

Now we are examining mathematics concepts learned through computer programming and find computer programming can be most supportive of the above considerations. Let me illustrate with the following problem:

Write a computer program
in BASIC to test whether
or not a set and a binary
operation form a group.

As students work on such a problem they must (may): 1) identify concepts such as: group, set, binary operation, function, well-defined, into, domain, range, Cartesian product, associativity, identity element, inverses; 2) specify the universe over which the concept is defined, for example, without loss of generality, restrict the sets to finite sets of the first n natural numbers, or test each potential identity element with every element of the set; 3) identify relevant and irrelevant features, for example, it does not matter that all sets of cardinality three are represented by the set $\{1,2,3\}$ and the natural number properties of the elements 1,2, and 3 are irrelevant to our purposes here; 4) identify logical connectives needed to define the concept, for example, for a group, the connectives are conjunctive and the computer program, to be correct, must reflect all the conjunctions; 5) include negative instances as part of any learning activities, and 6) test knowledge of concepts with new instances sampled from the universe, for example, to test the effectiveness of the program a broad range of both positive and negative instances must be sampled from the universe.

Here are some interesting discoveries of writers of such programs:

1. Consider the following output:

	1	2	3
1	1	2	2
2	2	1	1
3	3	1	1

Not associative for triples 223, 233, 322, 332

Identity = 1

Inverses 1 : 1
 2 : 2, 3
 3 : 2, 3

Question: why are the inverses not unique?

A careful review of the uniqueness proofs reveals uniqueness of inverses requires associativity, but uniqueness of identity elements does not.

2. Randomly generating tables of order 2, 3, and 4 and checking for associativity, identity element and inverses reveals the following frequency distribution of the relevant concepts in the universe:

	ASSOC.	IDENT	INVERSE
n = 200, order = 2	97	44	7
n = 200, order = 3	2	4	0
n = 200, order = 4	0	0	0

It is clear for tables of order 3 or larger, groups are a very rare bird.

Students writing such programs encounter an entirely appropriate set of experiences which are well supported by research in concept learning. Our plan is to systematically investigate the use of computer programming to teach fundamental concepts of mathematics.

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J. PROBLEM SOLVING / RESOLUTION DE PROBLEMES

WORD PROBLEM SOLVING: DIAGNOSIS AND TREATMENT

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RESUME

Les résultats d'une étude visant le diagnostic des difficultés dans la résolution de problèmes à texte (word problem) et d'une étude impliquant des méthodes pour remédier à des difficultés au niveau de l'interprétation des problèmes sont ici présentés. La description d'un troisième projet d'étude est aussi incluse. Dans la première étude des tests ont été construits à partir d'une série de manuels et administrés à environ 250 étudiants de sixième année. Ces tests ont permis d'établir un profil des résultats de chaque étudiant. L'interprétation de ces profils a permis de situer, pour chacun(e), la source principale de difficulté à résoudre des problèmes à texte au niveau de l'une ou l'autre des habiletés suivantes: calcul, lecture, interprétation du problème ou intégration c'est-à-dire la capacité de rassembler en un tout les habiletés distinctes qui précèdent pour arriver à résoudre un problème.

Dans la seconde étude, la procédure de diagnostic, précédemment élaborée, a été utilisée pour identifier les étudiants dont la principale source de difficulté était l'interprétation du problème. Ces étudiants, partagés en deux groupes, ont tous reçus une série de cours où les mêmes problèmes à texte étaient utilisés, la seule différence étant que les sujets du groupe expérimental étaient entraînés à choisir l'interprétation juste d'un problème parmi plusieurs interprétations possibles qui leur étaient présentées. Les progrès moyens de l'ensemble des étudiants ont été jugés significatifs mais non la différence entre les moyennes des deux groupes.

Une troisième étude est projetée afin d'élaborer une méthode de diagnostic plus efficace. Une procédure semblable à celle déjà décrite et s'appuyant sur un test standardisé sera développée pour permettre aux enseignants de diagnostiquer les difficultés à l'aide d'un moins grand nombre de tests. Si des résultats cohérents avec les conclusions précédentes, sont ainsi obtenus, cette méthode permettra une économie de temps et d'efforts pour les maîtres.

This is a report of the results of two studies on word problem solving and a description of a third study planned to answer a question revealed by the first two studies. The purpose of the two completed studies was to develop and test diagnostic and treatment procedures for middle school students experiencing difficulties with word problems.

Studies of word problem solving appear to be justified both by the difficulty and by the importance of word problems. Word problem solving has a history of being difficult. Word problem solving is difficult for teachers to teach and difficult for students to learn. Examine the work of students, listen to the opinions of teachers, and read the objective data from the National Assessment of Educational Progress for corroboration. In the latter, students were given some exercises requiring exactly the same computation as was required in word problems in another part of the test, and the finding was that students did better on the pure computation than they did on the same computation required in the context of word problems (National Assessment of Educational Progress, 1979, p. 14).

From the recent concern over the teaching of basic skills, problem solving has emerged as a basic skill in its own right (National Council of Supervisors of Mathematics, 1978). What good is it to master all the computational skills if one does not learn how and when to apply them? Word problems provide one way to help students develop an understanding of the contextual use of computational skills.

STUDY ONE: DIAGNOSING STRENGTHS AND WEAKNESSES OF SIXTH-GRADE STUDENTS IN SOLVING WORD PROBLEMS

Study of the literature led the investigators to believe that word problem solving skills can be classified into four broad categories:

1. Computation
2. Reading

3. Problem interpretation
4. Integration of the above three skills
into complete word problem solving

Computation and reading are widely recognized as crucial factors in solving word problems (Balow, 1964; Cohen and Stover, 1981; Glennon and Callahan, 1968, pp. 94-95; West, 1977). Many discussions on word problem solving seem to begin and end with these two rather obvious skill categories. However, a third category, problem interpretation, also gets a considerable amount of attention (Lerch and Hamilton, 1966/1970; Kantowski, 1980). Problem interpretation in the present study is considered to be the ability to set the problem up correctly. This includes choosing the correct operation and, if more than one operation is required, choosing the correct order of operations. Factors necessary for success in solving word problems are named and sub-divided in various ways in the literature. However, most writers on word problem solving include computation, reading and problem interpretation in some form as skill areas needed for solving word problems. The fourth skill area included in the present study, integration, seems to be missing from the literature on word problem solving.

Research questions arising from the above stipulated classification scheme are:

1. Can the developed procedure be used to answer for each individual student in the study these two questions:
 - a. At what level of difficulty can this student now solve word problems?
 - b. What is the main source of difficulty preventing this student from solving word problems at the next higher level?
2. What percentage of the students in this study have as their main source of diffi-

culty each of the four skill
categories named above?

A diagnostic procedure was developed to determine the relative strengths and weaknesses of youngsters regarding the four skill areas named above. A group of tests were constructed from a textbook series, and these tests were administered to approximately 250 sixth grade students. The method of construction of these tests and the analysis of the results enabled relative strengths and weaknesses to be determined and made it possible to identify one of the four stipulated categories of skills as the major immediate cause of problem solving difficulty for each student in the study. This diagnostic procedure is fully described elsewhere (Ballew and Cunningham, 1982a).

Complete data were obtained for 217 students. A profile of test scores was established for each student for whom complete data were available. Each profile consisted of these four scores:

1. Computation
2. Problem interpretation when the problems were read to the students
3. Problem interpretation when the problems were read by the students
4. Total problem solving

Analysis of the profiles led to the following findings in terms of the percentages of students falling into each skill category as their greatest immediate need:

- | | |
|---|-------|
| 1. Computation | : 26% |
| 2. Reading | : 29% |
| 3. Problem interpretation | : 19% |
| 4. Integration of the
above three skills into
complete word problem solving | : 26% |

These results show that no single skill area can be designated as the reason students cannot do word problems better than they do. Reading turned out to be the biggest area of weakness in terms of word problem solving for 29% of the students. This figure attains even more importance in light of the fact that students were placed in this category only if they could set up problems better when the problems were read aloud to them than they could when they had to read the problems for themselves. Analysis of the profiles also showed that being good in the separate skill areas involved in word problem solving does not ensure success in total word problem solving. Of the students in this study, 26% could not work word problems at as high a level as they could compute, interpret problems, and read-and-interpret problems when these skills were measured separately.

STUDY TWO: DIFFERENTIATED TREATMENT OF SIXTH GRADERS' WORD PROBLEM DIFFICULTIES

This study grew out of STUDY ONE, reported on above, in which the investigators developed and field-tested a diagnostic procedure for determining the main difficulty a student is having with word problems. In the present study the previously tested diagnostic procedure was used to identify those students whose major source of difficulty with word problems was in the problem interpretation area. The purpose was to see if children so identified could be helped through a planned series of lessons focussing on selecting the correct operation.

Details of the design and findings can be obtained from the investigators (Ballew and Cunningham, 1982b). A brief summary will be presented here.

Each of the investigators taught one experimental and one placebo group of fifth and sixth graders. Each of the students in the study had been identified as having his or her main difficulty in word problem solving in the area of problem interpretation. All groups

were taught exactly the same word problems over a period of six consecutive school days. The difference was that in the experimental groups emphasis was placed on selecting the correct interpretation for each problem from several possible interpretations provided to the students. The placebo groups worked the same problems with typical explanations and question-answering by the instructor. Pre-test and post-test mean scores, out of a possible score of 17, were:

	All Students	Experimental Group	Placebo Group
Pre-test	11.22	12.00	10.50
Post-test	12.44	13.00	11.93
Mean Change	+ 1.22	+ 1.00	+ 1.43

The mean gain for all students was found to be statistically significant ($p < 0.0022$). This led the investigators to conclude that the students in the study benefitted from the practice they had in solving word problems during the six lessons provided.

The difference between the mean gain scores of the experimental and the placebo groups was not statistically significant. The figures show that the experimental groups started with a higher mean score and ended with a higher mean score. The placebo groups started lower and thus had more room for improvement. Since this may have had some bearing on the outcome, it still may be that practice combined with special instruction in problem interpretation is helpful to those students classified as having problem interpretation as their main source of difficulty in solving word problems. The findings did not, however, support the experimental treatment as being superior.

STUDY THREE: THE USE OF A STANDARDIZED TEST
IN DIAGNOSING WORD PROBLEM SOLVING DIFFICULTIES

The purpose of this study is to develop a more efficient procedure for diagnosing word problem solving deficiencies based on a widely used standardized test. The need for this study arose from the two studies described earlier in this paper. The diagnostic procedure in both these studies involved administering a series of grade-level tests. Students who passed the grade-level test administered first were given the next higher test and those who failed the first test were given the next lower test until a level was established for each student, for each of four categories of difficulties. In working with teachers in workshops designed to instruct them in the use of this diagnostic procedure, it became apparent that a shorter, more efficient procedure is needed. Therefore, it was decided to develop and test a procedure based on a standardized test. If this can be accomplished, then a level can be established for each student in each of the four categories of difficulties with just one administration as opposed to the several administrations required in the previously developed procedure. If results consistent with previous findings can be obtained in this way, this will represent a considerable saving of time and effort for the classroom teacher.

If the results from the two procedures differ markedly, it will be of interest to see precisely what kinds of differences exist. For example, in both of the previous studies, fewer students were diagnosed as having their major source of difficulty in the problem interpretation area than in any of the other three areas. It would be helpful to know whether this same relationship would be found when using a standardized test. If not, it might be that the kinds of word problems on standardized tests do not require the same kind or level of problem interpretation as those in the textbooks used in the previous studies. If that is indeed the case, teachers need to be aware that the word problems in currently used textbooks may not provide adequate preparation for the standardized tests students take, and supplementary materials may need to be used to improve the teaching of word problem solving (Cunningham and Ballew, 1983).

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USE OF ANALOGIES AND SPATIAL TRANSFORMATIONS BY EXPERTS
IN SOLVING MATHEMATICS PROBLEMS

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ABSTRACT

Since Polya, Wertheimer, and Hadamard's descriptions of qualitative reasoning strategies used by scientists and mathematicians, very little data has been collected on whether these strategies are actually used by experts. This study used video-taped thinking-aloud interviews to examine the problem solving strategies of professors and advanced graduate students in technical fields. Evidence from these interviews documents the use of analogies, visual transformations, extreme cases, and other plausible reasoning strategies used by experts. In the case of analogies, it was found that there is more than one method for generating analogies, and that in addition, the process of critically evaluating the analogy is very important.

Considering helpful analogous cases and extreme cases, breaking problems into analyzable parts, and performing simplifying spatial transformations are key reasoning processes in solving non-trivial problems. These processes allow talented scientists to attack problems outside the domain of familiar problems for which they have established algorithmic procedures. They allow them to attack problems they have never seen before, giving them a degree of problem solving power and scope that is truly impressive. Previous reports [1, 2, 3, 4] have documented the fact that these qualitative reasoning processes are used by expert scientists in solving physics problems. This paper examines the possibility that empirical evidence for these processes can also be found in the case of experts' solutions to mathematics problems. I will first describe results from the earlier physics problem study, and then describe those from the current mathematical problem study.

EXPERT REASONING ON A PHYSICS PROBLEM

In the previous study, ten expert subjects were asked to solve the following problem.

Spring Coils Problem

A weight is hung on a spring. The original spring is replaced with a spring made of the same kind of wire, with the same number of coils, but with coils that are twice as wide in diameter. Will the spring stretch from its natural length, more, less, or the same amount under the same weight? (Assume the mass of the spring is negligible compared to the mass of the weight.) Why do you think so?

All subjects were advanced doctoral candidates or professors in technical fields. The study concentrated most on documenting and analyzing the use of analogies.

Some examples of analogies generated for this problem are as follows. One subject thought about a horizontal saw blade held fixed at one end and loaded with a weight at the other end. He felt that a long blade would bend more easily than a short one,

and this indicated to him that the wider spring might stretch more. Other examples of proposed analogies were that a longer horizontal "hairpin" shaped wire would extend more than a shorter one (see Fig. 1) and that a larger single "square coil" would stretch more than a smaller one.

Another subject examined the relationships between coil diameter, coiling angle, and wire length by thinking about mountain roads winding up narrow and wide mountains. The correct answer to the problem is that the wide spring will stretch farther (the stretch in fact increases with the cube of the diameter). This seems to correspond to most people's initial intuition about the problem. However, explaining why the wide spring stretches more (and explaining exactly where the stretch of the spring comes from), is a much more difficult task when taken seriously.

Some of the findings from this study were as follows.

- (1) Spontaneously generated analogies were observed to play a significant role in problem solutions of scientifically trained subjects. Seven of the ten subjects generated at least one salient analogy.
- (2) The subject generated a large variety of analogous cases. Not all of the analogies were to situations familiar to the subject. Some were novel cases in the form of Gedanken experiments that appeared to be invented by the subject.
- (3) In addition to the initial process of generating an analogy, there is a second process that is just as important in expert problem solving, that of critically evaluating the validity of the analogy.
- (4) Analysis of the transcripts indicated that there was more than one type of analogy generation method: two of these methods are the associative leap and the generative transformation.

The subject using an associative leap jumps to an analogous situation that differs in many ways from the original problem. For example, one subject compared the wide and narrow springs to two blocks of foam rubber, one made with large air bubbles and one made with small air bubbles in the foam. He had a strong intuition that the foam with large air bubbles would be easier to compress.

However, the associative leap was not the only analogy generation method observed. Another pattern was observed in which a subject

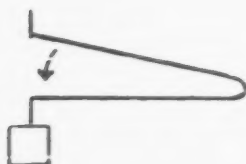


Figure 1



Figure 3

DOUGHNUT PROBLEM

Compute the volume of the torus (doughnut) below without taking an integral. Give an approximate answer if you cannot determine an exact one.

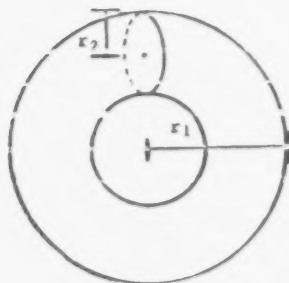


Figure 2

generates an analogy via a transformation which "warps" or changes the original situation A to produce the analogous situation B. These occur when a subject modifies an aspect of the original situation A that was previously assumed to be fixed. For example, some subjects "unrolled" the spring into a horizontal wire and thought about how much the wire would bend if the wire were held at one end and the weight were placed on the other end.

(5) Some subjects gave evidence of using spatial reasoning by referring spontaneously to imagining or picturing situations they were thinking about.

(6) Extreme cases such as considering a very narrow or very wide spring were observed as well. These were effective in adding confidence to a subject's prediction for the problem.

EXPERT REASONING ON A MATHEMATICS PROBLEM

The expert subjects who were asked to solve the spring problem also worked on the mathematics problem shown in Figure 2.

All subjects were advanced graduate students or professors in technical fields. This paper reports on preliminary results from five of the solutions to this problem and looks in detail at one of the solutions. We are only beginning to analyze these protocols, but already some behaviors parallel to those in the solutions of the spring problem are emerging. A common analogy generated for this problem was to consider the case of the "straightened out" torus in the shape of a cylinder. Subjects conjectured that the volume of these two objects might be the same. The condensed transcript excerpt S5 below gives one example of this approach.

- 001 S: ...Ok. So here's a doughnut. Now the question is how to get its volume. Uhh, the first thing that comes to mind is that it's probably pretty close to a worm, er, I mean a cylinder. Where you know, if you laid out the doughnut on the ground, uh, if you cut it open and laid it out, it would basically be the area of the base times the length around the middle. So let's see, I'll put down here number 1 is my first approximation which in fact may turn out to be the exact thing. Uh, I'll just turn it into a cylinder.
- 003 I: Mmm

- 004 S: In other words my hypothesis here is this volume of doughnut---and so that would-- πr_2 squared would be the bottom of the cylinder and then, uh, you know, I think the relevant length of the cylinder would not be r_1 but the distance uh, uh, to the middle there namely r_1 oops, r_1 minus r_2 --uh, right that-gets us to the middle, and then times 2 π ...
- 005 I: ...When you thought about the cylinder, do you know how that arose? Did it just sort of flash in your head?
- 006 S: Well I mean I in fact, er, this little r_2 was drawn here very nicely so I just imagine the knife cutting it open and you know, laying it out...
- 008 S: ...I mean to say I thought of cutting it at one edge and it sort of flopping down and then the uh, the doughnut becomes a cylinder.
- 009 I: Ok
- 010 S: And so I guess uh, you know with er, I don't think I've made any real algebraic mistakes here. I mean I think this er, probably I mean I feel pretty confident about that answer because what happens when you open the doughnut is that the, the top of the doughnut kind of expands a little. I mean, er, the part of the doughnut which was the inside stretches out a bit but then on the other hand, the part of the doughnut which was the outer perimeter gets crunched a little bit. And so probably those two things cancel. And you know, if the world is made correctly. But now I guess--
- 012 I: Just put a confidence number on it at this point and then you can go on.
- 013 S: Ok. I would say that on that I'm probably like 80% confident. Uhh, now what would happen if you did various things to the doughnut? Certainly you could argue that the answer gets closer and closer. I mean that this answer [the formula for a cylinder] is closer and closer to the correct one if uh, you know, if r_1 is much, much greater than r_2 , then in that limiting case, you've got to get this. Because that's just gonna be like you know it's gonna approach being a cylinder more and more. So whatever the correct answer is, it's got to have that [formula] as a limiting case if r_1 is much greater than r_2 ...
- 026 S: ...I suppose the other way you could imagine doing if you wanted to break it up would be to break it up into little wedges of doughnuts. So that if you were looking at it that way then you said ok, here's er, here's another
- 027 S: infinitesimal element which is a wedge like that, both faces of which have an area of uh, πr_2 squared and we'll call that now dA and again you know, it would boil down to that same equation again, if you added them all up

032 I: Yeah

033 S: into equation 2, right? And dZ essentially would be the midpoint, distance there. Ok. I mean I think I've er, I mean I think my confidence level at this point would be --like 95%.

The cylinder idea fits the definition of a spontaneous analogy as used here because it is a case which differs from the doughnut with respect to a feature (the shape of the doughnut) that is a fixed feature in the original problem. The observational definition of a spontaneous analogy used in both studies was the following:

(1) the subject, without provocation, considers another situation B where one or more features ordinarily assumed fixed in the original problem situation A are different; (2) the subject indicates that certain structural or functional relationships (as opposed to surface features alone) may be equivalent in A and B; and (3) the related case B is described at approximately the same level of abstraction as A.

The act of violating a feature previously assumed to be fixed is the creative aspect of producing an analogy. The difficulty of such acts is presumably the underlying source of Wertheimer's finding [5] that many students do not think to modify the shape of a parallelogram in order to compute its area.

RESULTS

Analogies. All 5 of the subjects spontaneously considered the analogy of the cylinder. Thus, spontaneous analogies were observed to occur in a number of expert solutions to a mathematics problem. Some subjects, such as S5, critically evaluated the analogy relation they had constructed by questioning whether the volume they had constructed was really the same as that of the torus. Earlier it was stated that the process of criticizing and evaluating an analogy is just as important as the process of generating it in solving science problems. This appears to be true in the case of mathematics problems as well. Subjects who think about an equivalent cylinder must choose a cylinder of the right length, and they often take pains to critically evaluate their choice of length.

For example, S5 above chooses the central or "average" circumference of the torus, $2\pi(r_1 - r_2)$, as the length of the cylinder. But he then evaluates this choice in lines 10 and 11 by giving a compensation argument about the inside stretching and the outer part getting "crunched". He also evaluates his prediction further by using an extreme case in line 13. If we call the circular curve passing through the centers of the circular cross sections of the torus its midline, then S5 seems to be implying that since the curvature of the midline gets less and less as r_1 gets very large, a local section of the torus will look more and more like a cylinder.

Analogy generation methods. A striking feature of this particular protocol is the explicit evidence for generating the analogy via a transformation rather than via an associative leap. The most explicit criterion used to code for a transformation is the subject referring to changing a fixed feature of the problem. Here the subject makes statements like: (line 2) "If you cut it open and laid it out...", and "...I'll just turn it into a cylinder...", referring explicitly to changing the shape. This method contrasts to an associative leap, where the subject is simply reminded of a familiar situation via a direct association (for example, if the subject were reminded of another problem he had seen about a torus). This protocol provides fairly explicit evidence for the possibility of generating an analogy via a transformation.

Spatial reasoning. The protocol also provides evidence for the role of spatial reasoning. First and most obviously there are references to spatial relations between objects that are primarily qualitative in nature, such as (line 2) "You know, if you laid out the doughnut on the ground"; and (line 11) "The part of the doughnut which was inside stretches out a little bit." Passages of this kind suggest that the subject is (1) imagining manipulating concrete or idealized objects, and (2) experiencing the anticipated outcomes of his manipulations via imagery.

Secondly, there are more explicit references to imagery. An imagery

report is defined as occurring when the subject refers to imagining, picturing, "remembering a diagram for", hearing, or "feeling what it's like to manipulate" a situation. We refer to a dynamic imagery report if the reference is to imagining a situation which does not remain fixed but changes with time. In this study we are concerned here with spontaneous imagery reports where the interviewer does not ask the subject whether an image was used.

Examples of dynamic imagery reports in the protocol are: (line 6) "I just imagine the knife cutting it open."; and (line 26) "You could imagine...if you wanted to...break it up into little wedges of doughnuts."

Thus it is possible to point to some evidence in protocols which supports the hypothesis that spatial reasoning involving imagery of a qualitative and dynamic nature is involved in expert problem solving.

"Creative cutting". The reference to cutting the doughnut into wedge-shaped pieces and computing their volumes documents the strategy of breaking a problem into parts--in this case the subject partitions the problem symmetrically into a number of equivalent parts. One could treat this as the simple application of a heuristic, but the trouble with the heuristic "break the problem into simpler parts" is that it does not tell you which parts to form. Such an act can require considerable creativity and ingenuity. (Other subjects have been observed to attempt other partitions as well, which will be discussed in a future paper.)

By speaking of an infinitesimal slice, S5 is in danger of breaching the request in the instructions to refrain from taking an integral. Indeed, creative cutting and partitioning of just this kind is an essential skill for applying the integral calculus to non-trivial situations. As in the case of analogies, the breaking into parts process is in effect an attempt to find a conserving transformation which leaves one with one or more simpler problems. It is

interesting to note that the wedges can be stacked alternately as shown in Fig. 3. In the limit, this can provide a connecting plausibility argument for the validity of the original analogy to a cylinder of length $2\pi(r_1 - r_2)$. The discovery of such conserving spatial transformations and equivalences can be a great source of satisfaction and appreciation for the interconnectedness of mathematical ideas. It was physical-spatial transformations of this type that apparently allowed Archimedes to develop many functional ideas underlying the integral calculus over two thousand years ago. (See description in [6].)

CONCLUSION

The ability to perform relevant spatial transformations, the ability to consider and evaluate analogous cases and extreme cases, and the ability to break problems into parts intelligently are crucial skills for solving non-trivial problems. The fact that experts use these processes can be documented in problem solving protocols, and the nature of the processes can be analyzed. In saying that these strategies played an important role in a number of the problem solutions we mean that they were involved in a serious attempt to understand or solve the problem and were not just proposed by the subjects as an ornamental side comment or as a check on a firm answer. We are currently investigating the abilities of students as well as experts to use these processes, and as we understand more about their nature, we should be able to design more effective instructional experiences which foster them.

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LEVELS OF LANGUAGE IN MATHEMATICAL PROBLEM SOLVING

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RESUME

Cet article suggère qu'un modèle réaliste de la compétence en résolution de problèmes mathématiques devrait comporter au moins quatre systèmes de langage de haut niveau: (a) un système pour le traitement verbal et syntaxique du langage "naturel", (b) un système pour le traitement "imagistique" (non verbal), (c) un système pour engendrer des mouvements à l'intérieur du langage de notation formelle et (d) un système pour la planification et le contrôle exécutif. Les produits de sortie (output) de l'un de ces systèmes peuvent servir de données d'entrée (input) pour n'importe lequel des autres. Le langage de planification tient lieu de métalangue en regard des autres, pouvant accepter leurs règles de procédure comme données et les modifier. De plus, le langage de planification a une capacité d'auto-référence, de sorte que des procédés heuristiques peuvent être employés pour agir sur le domaine des procédés heuristiques tout autant que sur les domaines des autres systèmes de langage. L'analyse théorique des variables de tâche, conjuguée à la recherche empirique, suggèrent que les langages décrits constituent le nombre minimum de systèmes de langage nécessaires pour simuler la résolution "intelligente" de problèmes. L'utilité de ce modèle pour améliorer l'enseignement de la résolution de problèmes mathématiques est aussi discutée avec des exemples d'activités dont les objectifs incluent l'échange d'information entre les quatre systèmes. La faculté d'auto-référence du langage de planification est illustré au moyen d'un processus heuristique "d'essai et erreur".

INTRODUCTION

The purpose of this paper is to elaborate on a model for mathematical problem solving based on four levels of language processing (Goldin, 1982). The intent of the model is to provide a unified framework within which competency in problem solving can be discussed, teaching objectives can be formulated, and observed problem-solving processes can be simulated. The model's origins lie in the conclusions of an exhaustive study of task variables in mathematical problem solving (Goldin & McClintock, 1979). Taking the viewpoint that the task functions as a measuring instrument in research, evoking the problem-solving processes which are to be recorded and studied, it becomes essential to understand how characteristics of the task affect these processes. The following main categories of task variables are observed to have such effects: (a) syntax variables, describing the grammatical structure and complexity of the problem statement, (b) content and context variables, describing respectively the mathematical and non-mathematical semantic content of the problem, (c) structure variables, describing the characteristics of formal problem state-space representations and algorithmic procedures, and (d) heuristic behavior variables, describing the heuristic processes which are logically applicable to the problem and the consequences of applying them. Much is known about which changes within these categories of task characteristics can affect problem-solving outcomes, and what effects are observed to occur. The present model seeks to address directly the question of how each family of task variables produces its observed effects. Consequently, there is a close association between the levels of language processing comprising the model, and the identified groupings of task variables.

The model is strongly motivated by information processing theories. Proceeding on the assumption that a meaningful simulation of the human being as an information processor should be based on higher-level languages, rather than the "machine language" level of neural nets, it attempts a preliminary description of those higher-level languages which might be needed. The term "language" is understood here in the broad sense of a system for processing symbol-configurations, not in its ev-

everyday meaning of a system for verbal intercourse. The model's framework is constructed with the eventual goal of accounting for a large portion of the experimental findings about problem solving in the psychological and educational literature: e.g. the roles of strategies and heuristic processes, "unconscious" processing, spatial visualization, pattern recognition, and "insight." With these considerations in mind, then, it is proposed that at least the following four higher-level language systems are entailed: (a) a system for verbal and syntactic processing of "natural" language, accounting for effects of task syntax variables, (b) a system for imagistic (non-verbal) processing, accounting for effects of task content and context variables, (c) a system for generating moves within formal notational languages, accounting for effects of task structure variables, and (d) a system for planning and executive control, accounting for effects of task heuristic behavior variables. Output from any one of these systems, appropriately transformed, should be able to serve as input for any of the others. Fig. 1 summarizes some of the relationships which exist among the language systems in this model.

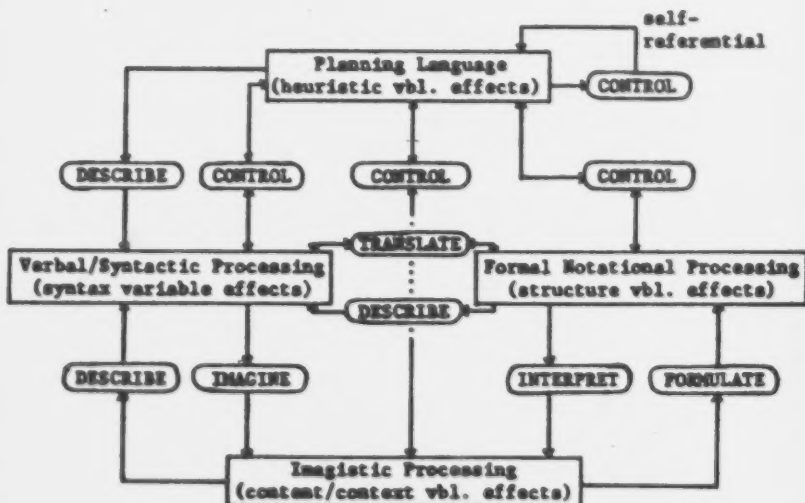


Figure 1. A Model for Competency in Mathematical Problem Solving.

Here I shall discuss briefly each language system. Before proceeding, it should be noted that many features of human information processing are omitted from this description--for example, the model does not address questions of how information is stored in memory and retrieved, nor does it include the role of "affect" in problem solving. These are possible elaborations of the theory in the future.

VERBAL AND SYNTACTIC PROCESSING

Problem solvers can process a problem's verbal statement in ordinary language such as French or English. At least some of this processing can be described on a purely verbal and syntactic level. Solvers interpret the grammar of sentences, and associate words and phrases with other words and phrases. Question sentences are distinguished from declarative statements, and information is sorted to some degree by virtue of syntax into that which is given and that which is desired. One can envision a computer programmed with dictionary information (definitions, synonyms, attribute lists, word associations, etc.) and grammatical rules (procedures for parsing sentences, obtaining referents of pronouns, etc.) which would permit considerable processing on this purely verbal level. Such processing could account for observed effects of manipulation of syntax variables, although up to now such effects have been studied mainly in linear regression models which often include effects due to other variables as well.

Let us consider inputs to and outputs from this system. Of course input can be a verbal statement communicated in print or orally. But solvers also have the capability to describe a non-verbal configuration in words. Here I call this process DESCRIBE, and envision it as a kind of subroutine whose input can be an imagistic, formal, or planning level configuration, and whose output is a verbal statement serving as input to the verbal/syntactic processing level. The configurations in the domain of DESCRIBE can be regarded as modeling those of which one is "conscious" during problem solving. One possible output of verbal/syntactic processing might be a formal notational configuration. That is, a learned procedure which I call TRANSLATE can accept as input a

verbal statement and output an arithmetic statement or algebraic equation. Many teachers seem to regard this as the main objective in solving "word problems," and the well-known STUDENT program of Bobrow (1968) models this idea very well. However, this process may well be the antithesis of what we mean by "insightful" problem solving. The pitfalls of such direct translation from the problem statement into mathematical notation include children's and teachers' frequent resort to "key words" as shortcuts for selecting arithmetic operations, without comprehension of the problem situation. An alternative procedure is to transform a verbal statement into an "imagistic" configuration. That is, a process which I have labeled IMAGINE can accept verbal/syntactic configurations as input, and can output non-verbal representations which are subject to imagistic processing.

IMAGISTIC PROCESSING

Sometimes almost involuntarily, problem solvers "imagine" a situation described by a verbal problem statement--"visualize" it, "feel" it, etc. The word "imagistic" that I have selected to stand for such non-verbal encoded configurations is not intended to suggest purely visual images, but to have the connotations of the word "imagine." There seem to be strong reasons for calling this level a language system, and including it as such in the model, though this modifies a perspective I took earlier (Goldin, 1982). There is abundant evidence in the psychological literature of the importance of factors relating to "spatial visualization" in mathematical ability. Problem solvers describe themselves as, for example, "mentally rotating" a figure. In the realm of imagistic processing one may also include pattern recognition, matching non-verbal sensory inputs with previously encoded information, one of the most powerful components of successful problem solving. Thus it appears that problem solvers process crucial information in ways that cannot be understood as purely verbal. On the other hand, such processing probably should not be relegated to a strictly "neural" level. A person may not be able to say "how" he or she "sees" a triangle when encountering the word, yet may be able to describe in detail the consequences of transforming the triangle in a variety of ways. The ini-

tial step of visualizing the triangle is undoubtedly very complicated on the neural level, but may be usefully regarded as a single step in a higher-level imagistic language.

One reason that imagistic processing is difficult to understand from the information processing viewpoint is that we do not know how imagistic configurations are encoded. It is not easy to describe the "visual image" that a person claims to have as a sequence of symbols, nor is it valid to assume that the brain functions as a miniature movie screen on which images are projected and manipulated. Pattern-recognition, furthermore, is a very elusive notion on the level of explaining how one manages to extract essential features from one configuration and identify them as "the same" as essential features from another. These are central questions for study.

Imagistic processing is conceptualized as that system whose operation accounts for the effects of manipulating task content and context variables. It is an interesting conjecture that "genius" in mathematics or physics has to do mainly with the imagistic level. It seems unlikely that the unusually great mathematician achieves that greatness merely by superior encoding of verbal problem statements in formal symbols, or by more efficient processing of formal notations, or by much better and more sophisticated heuristic plans. It is more plausible that the innovator has somehow succeeded in constructing a superior imagistic (non-verbal) representation, within which mathematical relationships inaccessible to others can be "seen." The challenge of mathematics for such a person is to develop appropriate notations within which such relationships can be precisely described and axiomatized, so that theorems can be proven.

FORMAL NOTATIONAL PROCESSING

Notations available for mathematical problem solving are highly structured formal systems. Learning their use probably comprises more than 95% of present-day school mathematics. Once a problem has been translated into such a notational language, purely formal steps can solve

it, though it may still be necessary to employ sophisticated search strategies (state-space search algorithms, etc.) in order to decide effectively which steps to take. Formal notational processing should be able to account for the observed effects of task structure variables. The present model suggests the importance of translation from imagistic configurations to formal configurations (by means of a process labeled FORMULATE in Fig. 1), and of using imagistic processing to motivate steps in formal notations. Thus, not only should the writing of equations for a verbal problem be influenced by the solver's visualization of the problem, but also the solving of the equations should often be done with reference to the meanings of steps taken on the imagistic level (by means of the process called INTERPRET in Fig. 1).

PLANNING LANGUAGE

Planning language includes heuristic processes used by problem solvers. It stands as a metalanguage with respect to the other languages, in that it not only controls the choice of steps within them, but can also modify their rules of procedure; the process labeled CONTROL in Fig. 1 is envisioned as carrying out these two functions. The effects of heuristic behavior variables characterizing problem tasks are to be accounted for by planning language processing.

The main point I wish to emphasize here is that planning language also has a recursive capability, so that the heuristic processes which comprise it can act not only on domains from the other language systems, but also on the domain of heuristic processes. For example, a student may wish to solve a word problem in algebra. First she attempts the use of "trial and error," substituting values for the variables and trying to successively approximate the solution. Evaluating this method as unsuccessful after several trials (perhaps the solution involves fractions), she decides to write algebraic equations and solve them formally. On the domains of formal and imagistic processing, the student first use trial and error, then equation-solving; but on the domain of planning language, the student also used trial and error, each trial being itself a heuristic process! It may be conjectured that

the immense diversity observed in heuristic problem-solving can be understood by means of relatively few processes, together with the capability of each process for acting on the others. This brings into play all of the beautiful complexities of "level-crossing" and self-reference discussed, for example, by Hofstadter (1979). Resolved also is the problem of avoiding an infinite regress on the executive level, so that higher levels of control are not needed to explain planning steps.

EDUCATIONAL IMPLICATIONS

The model's structure has intentionally been kept simple enough that its main features can be communicated to teachers. The next step is to develop teaching objectives with explicit attention to all four of the language systems we have used to model problem-solving competency, and to exchange of information among them. While most school mathematics instruction is now devoted to the notational level, and some to the verbal level, little is devoted to heuristic planning or imagistic processing. Perhaps many of the failures of the "new math" curricula can be understood as caused by misunderstanding (by teachers or authors) of imagistic and heuristic objectives intended by mathematicians. For example, in the teaching of bases other than ten, a worthwhile mathematical objective could be improved visualization of the meaning of place value (imagistic level); another could be improved ability to discuss, compare, and modify conventionally adopted notation in mathematics (planning level). Few would have argued for a wide need to teach new mechanical algorithms for computation in various bases or for conversion between other bases and base ten (formal level). Perhaps predictably, algorithmic objectives tended to supersede the imagistic or heuristic objectives, since the latter were rarely made explicit. In consequence, much of the value of new curriculum content has been lost.

We thus need to formulate problem-solving activities suitable for stating and achieving teaching objectives at all four language levels. Space permits but one example. In a well-known problem, "You are standing at the bank of a river with two pails. The first holds exactly three gallons, and the second exactly five. The pails are not marked

for measurement in any other way. Find a way to carry exactly four gallons of water away from the river." Here the first stated objective should be construction of a non-verbal representation of the problem situation, and manipulation of it. Some children may experience anxiety when the problem is posed, since no previously taught algorithm applies. It may therefore be helpful to postpone posing the goal, and instead describe the problem situation and ask, "What can you do? What do you see?" Children may be encouraged to imagine pouring water from pail to pail, and asked about each step, "What happens if you do that?" Once a firm imagistic representation has been constructed, the goal of obtaining exactly four gallons can be introduced. Some students may generate many mathematical steps through visualization, but the next objective should be construction of a formal representation. Children may be guided to represent a situation by a pair of numbers, and move back and forth between the imagistic level and the formal notation as steps are taken. Achievement of the goal state through standard "working forward" techniques can be a third objective, but the activities should not end there. On the planning level, objectives can be formulated and achieved which include the search for multiple solutions (this problem has two distinct solution paths), working backward, and problem generalization (discovering the possibility of obtaining a whole class of problems by changing the capacities of the pails). It is my hope that the model described here will help teachers employ such activities in a meaningful, objective-oriented way.

ACKNOWLEDGMENTS

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THE ANALYSIS OF A HEURISTIC PROCESS: "THINK OF A SIMPLER PROBLEM"

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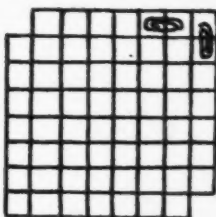
RESUME

Nous pouvons noter un certain progrès dans l'analyse du processus heuristique, "pense à un problème plus simple (TSP)" (Think of a simple problem) en ce qui concerne: (1) les raisons possibles à priori de penser à un problème plus simple (2) les avenues possibles pour rendre plus simple un problème mathématique (3) les domaines possibles d'application du processus heuristique TSP et (4) les stratégies prescriptibles suggérant quand penser à un problème plus simple. Des informations provenant des protocoles individuels facilitent l'analyse. Ces dimensions procurent un cadre pouvant être utile à la discussion des processus heuristiques en général. Une attention particulière est apportée à deux façons permettant de simplifier un problème - en attribuant une valeur entière plus petite à un paramètre ou en réduisant le nombre de variables. Des suggestions sont faites pour la formulation de sous-processus de TSP en tant qu'objectifs d'enseignement et pour des façons de mesurer la compétence des élèves à utiliser ces sous-processus. L'analyse démontre aussi certaines des possibilités d'un "langage de planification" dans un modèle de résolution de problème en mathématique, modèle proposé par Goldin qui inclut à capacité récursive ou d'auto-référence des processus heuristiques.

This study suggests a framework for analysis of competency in heuristic processes based on the following dimensions: (1) advance planning reasons the solver may have for using the process; (2) ways in which the process can be employed; (3) domains to which the process may be applied; and (4) prescriptive strategies (Schoenfeld, 1979a;1980) suggesting when the process should be used. In the following discussion this framework is exemplified for the heuristic process, "think of a simpler problem" (TSP), thereby illustrating some features of "planning language" in a proposed mathematical problem solving model (Goldin, 1982; 1983). Teaching implications are also explored.

PLANNING REASONS FOR EMPLOYING TSP

The following are some of the reasons a solver may have for thinking of a simpler problem in mathematics, at the time of the decision to do so: (a) The solver plans to make use of the answer to the simpler problem in solving the original problem. For example, "Find all real values of x such that $3^{2x} - 3^x = 72$." A simpler problem results from the substitution $y = 3^x$, whose answer is used to solve the original. (b) The solver plans to imitate the method that works for the simpler problem. For example, "A paper clip may be placed so as to cover two squares adjacent horizontally or vertically, as shown in the diagram.



Can all the squares be covered by paper clips? If so, how; if not, why not?" A simpler problem is obtained by considering the smaller configuration to the right (for which the outcome of a trial has been illustrated). The solver may plan to imitate either a method of covering all the



squares (should this be found in the simpler problem), or a method of demonstrating impossibility. (c) The solver plans to generate a series of simpler problems, and to look for a pattern in either their answers or their solution methods. For example, "The diagram (next page) shows a triangular grid six units in length on each side. How many triangles one unit on a side are there in a grid that is 20 units on each side?" The solver may hope that the series



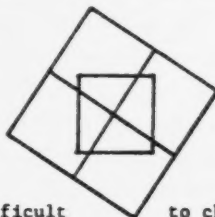
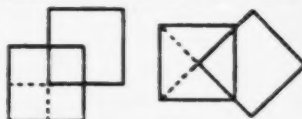
of simpler problems obtained from grids 1, 2, 3, 4, ... units per side will reveal a pattern of answers or methods leading to solution of the original problem.

(d) The solver hopes that construction and solution of a simpler problem or problems will assist with the original problem through some kind of improved perspective or "change of Gestalt," even apart from answers obtained or methods employed. For example, "Two squares of length s on a side are placed so that a corner of one square lies at the center of the other, as shown. Find the



area of the region where the squares overlap in terms of s and the indicated relative angle θ ." In considering the simpler problems where $\theta = 90^\circ$ or 45° , the solver may plan not on using the answers or methods

directly, but on finding a better way to "perceive" the original problem. One way in which this might come about is through detection of the symmetry that is present, so that the simpler problem diagrams to the right suggest that the original problem be viewed from the perspective of either the first diagram below (Schoenfeld, 1979b) or the second. These



directly imply the answer, $s^2/4$, independent of θ . (e) Finally, the solver may have no particular advance planning reason for using TSP, beyond the hope that it will somehow be helpful. This

situation is difficult to characterize by means of a problem example, because the eventual helpfulness of TSP is always by some specific means. We nevertheless distinguish this situation from the others, keeping open the possibility that the solver decides to use TSP on "general principles," without a more detailed plan in mind.

The above planning reasons for employing TSP can be understood in terms of the self-referential feature of planning language (Goldin, 1983). In cases (a)-(d), the solver either (1) has a heuristic plan which "calls" TSP as a "subroutine" during its execution, or (2) by

virtue of TSP itself, early in its execution, evokes a heuristic plan toward which TSP contributes. Thus TSP is not the only "operative" heuristic process. Even in case (e), TSP may be functioning as one trial in a "trial and error" process, in which each trial is itself a particular heuristic procedure.

WAYS OF EMPLOYING TSP

In our view the ability to simplify a mathematical problem in various ways generally requires guided experiences with examples. It is less common for students, through the exercise of other heuristic processes, to "discover" wholly new ways of simplifying problems. Thus the ways of employing TSP may be considered mainly "domain-specific" knowledge. The following list is preliminary, and not intended to be exhaustive:

(a) A smaller number can be substituted for a larger number. (b) A value can be substituted for a variable. (c) An analogous problem with fewer variables can be devised. (d) An analogous problem with fewer dimensions can be devised. (e) An extreme case of the problem can be posed. (f) A symmetrical case of the problem can be posed. (g) A subgoal of the original problem can be decided upon. (h) A "change of variable" can be made. (i) A geometric transformation can be performed. (j) The problem can be restated to apply to a more narrowly defined situation. (k) A given condition in the problem statement can be dropped. (l) The notation adopted can be simplified.

Particular ways of simplifying mathematical problems are frequently associated with particular "planning reasons," but the association is not universally necessary. For example, "choosing a subgoal" (g) might be associated with planning to make use of the answer to the resulting simpler problem; however, one might also plan to imitate the method of reaching the subgoal, or hope to find a helpful change in perspective. We also remark that (b), (e) and (f) in the above list are special cases of the heuristic process, "try special cases." For some of the methods, especially (c) and (d), a crucial step may be verification that a conjectured simpler problem is indeed "analogous" to the original. Our ongoing study of TSP employs tasks to which (a) and (c) apply.

THE DOMAINS OF TSP

In characterizing the domains of TSP, we again stress the self-referential capabilities of heuristic processes. (a) The process may be applied to the original problem P, yielding the simpler problem P'; we write, $P' = TS(P)$. (b) The process may be applied recursively to P', so that $P'' = TS(P') = TS(TS(P))$. For example, let P be the problem, "Show that for positive integers p, q, r, s, $\frac{(p^2+1)(q^2+1)(r^2+1)(s^2+1)}{pqrs} \geq 16$." Then $P' = TS(P)$ might be, "Show that for positive integers p, q, $\frac{(p^2+1)(q^2+1)}{pq} \geq 4$," and $P'' = TS(P')$ might be, "Show that for a positive integer p, $(p^2+1)/p \geq 2$." (c) The process may be applied to the derived problem Q, "Apply heuristic process H to P," for any process H (including TSP itself); we write, $Q = Q(H, P)$ and $Q' = TS(Q)$. Since this becomes rather complicated, it is worth elaborating. Given a problem P, the solver wishes to use TSP, and is thus faced with the new problem Q which we paraphrase, "Think of a problem simpler than P but related to it." Suppose this proves difficult. Taking $TS(Q)$, the solver may obtain Q', "Think of a problem not necessarily simpler than P but related to it." Q' is obtained as $TS(Q)$ by dropping one of the conditions of Q, method (k) of the preceding section. Alternatively, the solver may try to apply a heuristic process H such as "draw a diagram," and thus face the new problem $Q(H, P)$, "Draw a diagram of P." If this proves difficult, the solver may try to "think of a problem simpler than, 'Draw a diagram of P,'" and come up with Q', "Draw a diagram of P'," where P' is a problem simpler than P. In formal notation, $Q' = TS(Q(H, P)) = Q(H, TS(P))$. The possibilities for heuristic processes acting on each other in this fashion are endless.

PRESCRIPTIVE STRATEGIES

Schoenfeld (1980) remarks that knowing how to use a heuristic process does not imply knowing when to use it. However, delineation of ways in which a process can be used permits the formulation of prescriptive criteria for application of the process. For example, substitution of a smaller number for a larger one (as a means of obtaining a simpler problem) immediately suggests, "try TSP on problems having large integer pa-

rameters," while finding analogous problems with fewer variables suggests, "try TSP on problems with many variables (especially when they enter symmetrically)." In general, one might conjecture that TSP should be tried whenever one or more of the ways of simplifying a problem succeed readily in generating a simpler one. Prescriptive criteria can then be developed by making explicit the conditions under which this is likely to be true. On the other hand, such criteria do not exhaust the worthwhile prescriptive strategies for TSP; additional possibilities are, "try TSP when it is difficult to keep track of information gained while attempting the original problem," or "try TSP when no other procedure comes to mind." Such broad "default options" sound quite trivial—yet so many students are willing to abandon problems, even after much struggle, without ever attempting TSP. This situation is illustrated by the protocol which follows.

EXCERPTS FROM A TSP PROTOCOL

Our analysis of TSP has been influenced by our combined experiences with students, and by the study of individual protocols. The following abridged protocol is that of S, a male high school senior, age 17.

"Show that for positive integers p, q, r, s , $\frac{(p^2+1)(q^2+1)(r^2+1)(s^2+1)}{pqrs} \geq 16$."

(S) [Reads problem] Wow, there's probably a quick way to do this problem, but then sometimes I think there is and I try it and I turn out to be wrong. I'm just going to do this the long way and see what happens. Okay, I'm going to multiply the top [lengthy, correct calculations] ... Boy, this is getting messy [further correct calculations] ... and I don't think this is going to help at all ... Let me see how this works. Suppose $p = q = r = s = 1$. Then I get 16 over 1 ≥ 16 . Okay, that's true. Suppose I use zero ... oh, but p, q, r, s must be positive integers. Okay, so can't use zero. Let me multiply by $pqrs$, so $(p^2+1)(q^2+1)(r^2+1)(s^2+1) \geq 16pqrs$. That doesn't seem to help either. I don't know. I can't do these problems. [Stops] ... (I: Suppose I asked you to think of a simpler problem?) (S) Well, it might. What I might do is just start with two numbers p and q and try to show that, let's see, all right, let me see what I can do with that. So we're going to take two positive integers p and q and try to show that $(p^2+1)(q^2+1)$ over pq , now I don't think that I can go so far as to say that it would be greater than 16, because if I prove that, it may not be able to help me into proving these. Maybe because I'm taking two numbers and multiplying the top and bottom by two less factors, I should take the square root of 16, but then again I'm not sure. I don't know if these is the right number to be working with, being that I'm wor-

king with a simpler case. Let's see [decides on "4"]. Now again if I square out [writes $(p^2q^2 + p^2 + q^2 + 1)/pq \geq 4$] I find that I can't simplify this either. No, I don't think I can. Well, if I did divide the denominator into the numerator, I would get this $[pq + p/q + q/p + 1 \geq 4]$, actually, that's the same thing. No, I don't know. I just can't put my finger on how to do these problems. How to do this problem even in the simple case. [Stops] (I: Did you take the simplest case possible?) (S) No. Actually I didn't. I guess one number would be the simplest case. ... Okay. Just one number, a positive number over p. Okay, let me see what this has to equal. If the number is greater than zero we are trying to see what that $[(p^2+1)/p]$ has to be greater than or equal to. Maybe I have to show that this is greater than 2. I'm not sure, let me just see. Oh, yeah, all right, so this is going to work. If $p = 1$, then p^2+1 is greater than 2, and if $p = 2$, I'll get $5/2$, all right. This will work for all $p \geq 1$. Now let me try some fractions [calculates correctly for $p = 1/2$ and $p = 1/4$] ... All right, I think that I'll be able to demonstrate but not prove that for one number it holds true. Oh, wow, now I see. I see. Now if p^2+1 over p holds true, then I can say for another positive integer, we call that one q, $(q^2+1)/q \geq 2$ by the same demonstration. And therefore, if I were to multiply these two fractions I would get this. This idea here ≥ 4 and this is really great. Now I can take another positive integer r, and say what holds true and another positive integer, and that way I can prove this. (I: Nice.) (S) Okay, that was good. I never, I just never thought of it but at the prompting ... then again I'm not sure this is proof any more, but this is just an actual demonstration of a trial and error kind of thing. But this would actually be p^2+1 over p. I could probably show that. I'm pretty sure I could. I like this problem now. I didn't like it before.

To us the most interesting features of this protocol are: (1) the clear separation between S's competency in how to use TSP (he obtains problems with fewer variables, and he correctly checks that they are analogous), and his lack of a prescriptive criterion for when to do so; (2) his inability to apply TSP recursively without further prompting; (3) his patience with the processes of formal manipulation in algebra, and trial and error, which he uses repeatedly; (4) the dramatic effects of the investigator's suggestions, "Suppose I asked you to think of a simpler problem?" and, "Did you take the simplest case possible?" and (5) the remarkable change in affect after successful application of TSP, despite the fact that a portion of the problem remained unsolved.

TEACHING OBJECTIVES AND MEASUREMENT OF TSP SUBPROCESSES

A logical corollary of the analysis presented here is the idea of for-

mulating teaching objectives based on subprocesses of TSP. For example: (a) To "correctly" simplify a given problem when told to do so by substituting a smaller number for a larger number, by substituting a value for a variable, by devising an analogous problem with fewer variables, etc. (b) To decide on a suitable method for simplifying problems of various kinds and carry it out, when told to do so. (c) To decide without prompting to think of a simpler problem, when specified prescriptive criteria are met. (d) To state "planning reasons" for thinking of a simpler problem, explaining what outcomes are hoped for. (e) To apply TSP recursively. (f) To apply TSP to the domain of heuristic processes as well as to the domain of stated problems. Fundamental to the goal of achieving such teaching objectives is the ability to measure competency in TSP subprocesses. This ability is also important to researchers. It is our belief that test questions can be formulated which accomplish such measurements, and therefore we close this paper with some examples: (a) Eve is trying to find the last two digits of $11^{20} - 1$. She decides to look at the simpler problems, $11^1 - 1$, $11^2 - 1$, and $11^3 - 1$. What is she hoping to accomplish by this? [tests for planning reason, "looking for a pattern in answers or methods"] (b) Jerry wants the sum of the coefficients when $(x+1)^{31}$ is expanded. What simpler problems should he look at? [tests for substitution of smaller integer values for large value] (c) Bruce is trying to prove that for positive integers p, q, r, s , $(p^2+1)(q^2+1)(r^2+1)(s^2+1)/pqrs \geq 16$. He is considering the simpler problem, $(p^2+1)(q^2+1)/pq \geq 4$, but is still stuck. What should he do next? [tests for recursive use of TSP] We believe that teaching objectives such as these, with appropriate measures, can facilitate students' learning of TSP as a heuristic process.

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PHASES IN MODELING AND PHASE-RELATED PROCESSES

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RESUME

Ce texte tente de décrire plusieurs phases de la résolution de problèmes. Il importe que les phases (ou stades) soient définies en termes de leurs fonctions dans le processus de résolution de problèmes car, avons-nous trouvé, l'utilisation avec succès d'heuristiques et d'autres procédés dépend de la fonction qu'ils remplissent. Ce texte est une première tentative pour décrire comment les fonctions changent tout au long du processus de résolution de problème. Nous croyons que ces fonctions peuvent se décrire en termes d'applications entre le problème et les modèles cognitifs de celui qui cherche à le résoudre. Les phases que nous décrivons mettent l'accent sur les similitudes qualitatives des fonctions plutôt que sur les accroissements quantitatifs du bagage de connaissances.

The solutions of many types of problems traditionally have been characterized as a uni-directional process from givens to goals. A student's understanding of the problem increases quantitatively as they procede through Polya-type stages in a linear order: (1) understand the problem; (2) devise a plan; (3) carry out the plan; (4) look back. In contrast, we have found in the Applied Mathematical Problem Solving (AMPS) project that most solutions were characterized by qualitative shifts in the understanding of a problem, predominance of non-answer giving activities, and a non-linear progression through qualitatively different phases. There were many sessions in which students went through as many as 10 qualitatively distinct understandings during the solution process. We found that an overwhelming amount of time was spent on refining understandings about the problem. Once the problem was sufficiently clear, little time was needed to complete the procedures for finding the answers. We also found that the use of heuristics and other processes were not inately "good" or "bad". Rather, the beneficial and non-beneficial aspects were related to the funtions those processes served and not their placement in a linear progression. Therefore, we find it more useful to distinguish phases of problem solving in terms of their functional similarities rather than considering stages in terms of a linear progression.

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We have chosen to refer to phases in terms borrowed from Gestalt psychology because we want to emphasize the qualitative reorganization of understandings rather than quantitative increases in knowledge. Initially, we have identified three phases: interpretation, integration/differentiation, and verification. Interpretation is a series of mapping cycles that emphasizes mappings from the problem situation to cognitive models. Integration/differentiation is a series of mapping cycles that emphasize mappings within and among the cognitive models. Verification is a series of mapping cycles that emphasize mappings from the models back to the problem.

Past publications related to the AMPS project have described our problems, the student characteristics, solution processes and heuristics. In our presentation at the PME-NA Conference in Montreal, 1983, we will illustrate our concept of phases by focusing on one problem and a detailed analysis of one session. This paper will focus on what we mean by models and modeling, and then discuss phases of modeling.

MODELS AND MODELING

Models: Solvers create solutions by mapping problem situations to internal models. The solutions to our problems can seldom be described as a search from well defined givens to well defined goals. Instead, the process required input from the student to make decisions about what ideas, information, goals, and procedures were needed. Rather than linking together well established procedures, the solutions involved forming several, perhaps incomplete, under-

standings about the problem, and then usually refining those understandings. In this paper, we refer to students' understandings as conceptual "models". A "stable" model is internally consistent and well coordinated, while a "good" model minimizes conflicts between itself and the problem situation. Conceptual models are not static, but change when the problem solver explores new information or gains new insights.

Mapping Cycles: Students interpret a problem situation by mapping their perceptions to available conceptual models. There are three types of links in a single mapping cycle: a mapping over, a transformation within the model, and a mapping back.

Modeling: "Modeling" is the process of constructing and refining models through repeated mapping cycles. It involves integrating and differentiating different relevant models, or moving from one level of understanding to a qualitatively higher level of understanding. Early levels of a given model are characterized by distortions in which the student selectively notices only the most salient aspects of the problem and imposes characteristics on the problem that are not there. Gradually, as the student makes repeated mappings between the problem and the model, conflicts are reconciled and a stable, good model evolves. This spiraling model evolution is characterized by the occurrence of repeated mapping cycles while simultaneously the qualitative level of understanding increases.

PHASES OF MODELING

Three phases have been identified in our students' modeling behavior. The interpretation phase includes mapping cycles in which the mapping from the problem situation to available conceptual models serve the most important function of that mapping. Integration/differentiation includes collections of mapping cycles in which mappings within and among the solver's conceptual models serve the most important function. The verification phase includes mapping cycles in which mappings from the cognitive models to the problem situation is the most important function. The phases do not always occur in the order given. We have examples of sessions in which the students began by verbalizing a string of trial answers for tentative verification before taking time to seriously interpret the problem. In other sessions, students initially mapped to two or more inconsistent understandings of the problem before seriously considering the conflicts among those models.

The purpose of the interpretation phase is for the solver to become familiar with the problem. The student's manipulation of the problem materials in an unorganized, undirected manner allows for the evolution of many separate partial models that correspond to different aspects of the problem. The partial models usually represent only a portion of the problem, while simultaneously the same part of the problem may correspond with multiple contradictory models. It is the reconciliation of the partial models that the solver investigates during the integration/differentiation process.

Interpretation is manifested by a disorganized preoccupation with the problem materials, manipulating them in random, often illogical ways with only marginal interest in the results.

The function of the integration/differentiation phase is to form a stable conceptual model. Various relevant partial models are coordinated into one larger, stable structure through repeated mapping cycles within and among cognitive models. A single unified model is formed through a spiral evolutionary process in which the partial models are continually being refined, modified and combined. Behaviorally, this period is marked by relative lack of references to the problem materials. It is a period of quiet reflection and can occur unconsciously while the solver is consciously occupied with other tasks.

The function of the verification phase is to verify that the cognitive model is a good representation of the problem. This is not like checking an answer because a good model is not a solution to a problem, but a tool the solver uses to find a solution. During verification, the solver's model undergoes further spiral evolution through multiple mapping cycles between the model and the problem, but the changes in the model are most likely to be minor due to the predominance of the mapping from the model to the problem. Verification is characterized by a series of small organized experiments with the problem materials. The model predicts that certain relationships will hold and the solver tries a few test cases to see if the prediction is true.

CONCLUSIONS

We do not intend for our description of phases to imply a fairly simple categorization of functions. It is likely that we may change the number and names of the phases as our understanding of the problem solving process matures. Presently we are aware of many different levels at which phases seem to occur. As described in the major portion of the text, we see mapping cycles and modeling processes nested in phases. However, if a different perspective is taken, one can consider the three components of a single mapping cycle as parallel to the phases defined. Mapping over corresponds to interpretation, transformation within the model corresponds to integration/differentiation, and mapping back corresponds to verification. Another perspective reveals the concept of phases nested within phases. We have seen examples in which an apparently full cycle of phases results in a decision to abandon a model. Since the initial modeling provided information for the formation of a new model, the first group of phases can be thought of as nested in the interpretation phase of a new larger cycle.

Due to the complex nature of problem solving, we have described our phases in fairly tentative terms. Our purpose is to identify collections of similar functions that are "big enough" to bring to the consciousness of students. Since heuristics and other processes that are valuable in one phase may not be helpful in another, it is useful to link these processes to phases in terms of the functions

they serve. In this manner, phases may be able to serve as a prescriptive tool in instruction as well as a descriptive tool in problem solving.

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META-LEARNING IN WORD PROBLEM SOLVING
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RESUME

Cette étude vise à rendre explicite quelques-uns des procédés, habituellement cachés, utilisés dans la résolution de problèmes à texte (word problem). Pour y arriver, on a demandé à 60 élèves de sixième année de réécrire quatre problèmes pouvant être résolus en deux étapes, de façon à les rendre plus faciles à résoudre pour des élèves de troisième année. Les résultats montrent un grand nombre de changements qualitatifs dans la formulation des textes originaux. Un des changements typique consiste en ajout de détails contribuant à la vivacité, à la description dynamique des événements rapportés dans le texte. Ceci fut habituellement réalisé par l'addition d'un contexte, de justifications et d'adjectifs. Ces résultats appuient l'hypothèse de la présence de multiples considérations qualitatives dans le processus de compréhension de la description verbale d'une situation donnée. Cet aspect risque d'être négligé dans les problèmes conventionnels de l'école, problèmes qui ont une nature stéréotypée.

The pilot study reported here is intended to be an extension of a study published a few years ago under the title "The Stereotyped Nature of School Word Problems" (Nesher, 1980). It was suggested in that paper that problem solving consists of several processes, some of which involve quantitative considerations, but most of which involve qualitative considerations. Some of the processes touch upon surface structure transformations but others pertain to the underlying structure of the situation described to the child. These processes take place partly overtly but to a large extent covertly. This is why it is so difficult to discern the main source of children's (and adults') failure in problem solving. The present study tries to approach the covert processes by a methodology adapted from Stover (1979) as will be detailed later.

The 1980 model suggested two different types of underlying processes needed to solve elementary word problems. The one has to do with understanding the situation described by verbal means. This includes linguistic interpretation (on a lexical level and on the syntactic level), as well as constructing imaginarily the reference domain of objects and relations comprising the pragmatic context. The second has to do with aspects of mathematics as a formal language. This includes understanding which operations to choose, in what order to execute them etc.

It should be noted that this model emphasizes a stage usually neglected in instruction as well as in research. It is the stage in which the solver has to construct in his mind the entire situation and involves first of all a kind of a qualitative reasoning.

This is in the spirit of all research or inquiry. As T.S. Kuhn wrote: "One recurrent implication of the preceding discussion is that much qualitative research, both empirical and theoretical, is normally prerequisite to fruitful quantification of a given research field." (Kuhn, 1977 pp.213). Similarly if one reads recent research reports on experts' vs. novices' knowledge, one is

impressed with the fact that counter to our expectation, novices when facing a physical problem start immediately by manipulating all kinds of formulae and calculations, while the experts start with qualitative considerations. We were able to discern a part of the qualitative aspects involved in solving word problems through the experiment reported here.

The Experiment

60 sixth graders from a middle-class neighborhood were given four word-problems with the following request:

"Here are some problems which were difficult to solve for third graders. Can you please phrase them differently so that they will be easier for them. Thank you for your help".

After each problem, a space was left for the child to re-write his text. It took the sixth graders about 40 minutes to complete the re-phrasing of the four problems.

The Problems

All the problems had two-steps (i.e. three numbers were given and two binary operations were needed to solve them). This experiment did not cover all elementary two-step problems. If all combinations of the four operations (+, -, *, :) are taken into account, there are at least 16 types of problems (without manipulating parentheses). In our ongoing research we have a sample of about 250 word problems on a CAI (computer assisted instruction) program, representing all types of two-step problems. This supplies us with statistical data concerning the degree of difficulty of two-step problems. From this collection we have selected only four representative problems to serve in our pilot study. Another part of our study not reported here deals with direct protocol of third and fourth graders while trying to solve the same kinds of problems.

The four problems we have chosen are of the following types:

TABLE I

The Context	Needed Operation	Actual Numbers
1. crayons & boxes	$a + b = c$	$6 + 4 = 5$
2. fruits & baskets	$a = (b+c)$	$6 = (2+5)$
3. birthday, sweets	$c : (a+b)$	$30 : (8+2)$
4. boys & girls	$c - (a+b)$	$28 - (3+5)$

For the full text of the problems see Appendix A.

The actual four problems were selected to display information on the main variables upon which we focus. Research on one-step problems usually concentrated on the question, how does a child choose the correct operation when presented with a verbal text. The focus in the present study was different. Given that the child knows to choose the correct operation for a binary situation, how does he know in a more complex situation to order the operations that he has mastered separately? How does he know with what to start, what is the order of the two operations?

Our hypothesis was that this question is affected by several variables:

- 1) The main and the secondary operation.
- 2) The mathematical order of operations as defined by the field laws.
- 3) The order of given numerical information in the verbal formulation of the text.
- 4) The temporal order of events to which the text refers.

We were not able, of course, in a selection of four problems, to hold the above variables independent, but we wanted to get a 'feeling' for their effects. In what follows we will elaborate more on the above variables.

- 1) The main and secondary operation.

The main operation is the final operation whose completion gives the answer to the explicit question in the text. The secondary operation is the one that is done first in order to obtain prior information needed for the main operation. For example, our

problem #2:

Joseph has 6 baskets. In each of his baskets there are 3 apples and 2 bananas. How many pieces of fruit are there in Joseph's baskets?

In this case answering the explicit question means multiplying the number of pieces of fruit in each basket, by the number of baskets Joseph has (5*6). *) The multiplication of 5×6 is the main operation, but it could be performed only if $2+3$ the secondary operation, is first executed. The execution of the secondary operation ($2+3$ in our example) is done in order to answer an implicit question, not mentioned in the text, which is an important step in the solution of two-step problems. In this case the secondary (implicit) question is "How many pieces of fruit are in each basket?"

Our sample of problems included all possible operations as the main operations (+, -, *, :). The secondary operations were so selected as to be operations of another degree as will be explained in the next section.

2) The Mathematical Order

The order of execution of the operations in a given arithmetical phrase is well defined by the field laws. Children learn earlier that given a phrase with several operations of different degrees (+ & - defined as 1st degree; * & : as the 2nd degree; and power & roots as the 3rd degree), the higher degree operation is always performed first, unless otherwise indicated by the use of parentheses. Children also learn a few laws such as the commutative, associative and distributive laws which they can use in rearranging the given data. In our small sample all the problems involved operations of two different degrees in each problem.

3) The Order of Numerical Data

The letters a, b, and c, mark the order of appearance of the information in a given text, when "a" is always the first piece of information and "c" is the last. In two of our problems (#1 & #2) (see TABLE 1) the order of the numerical data was a, b, c. This was also the order of the written arithmetical sentence (i.e. $a \cdot b \cdot c$),

nevertheless the child had first to operate on b & c, before he could use the first piece of information (i.e. "a"), for the main operation. In two of our problems in the sample (#3 & #4), the last piece of information in the verbal formulation (i.e. "c"), has to be the first in the arithmetical sentence, but it is to be used last when the main operation is executed.

4) The Order of Events

Actually our sample did not represent this variable. Almost all the problems were non-temporal, and were of the type called "static".

Analysis of Findings

One should remember that the study reported here does not deal with the degree of difficulty of the various problems, nor with patterns of errors. These will be reported elsewhere. The concern here is only with children's reflection on the nature of the task of two-stage problems. The analysis, therefore, should pertain to this reflection. The data to be analysed is the re-written problems that the sixth graders prepared for the third graders, and the categories we use are derived from these texts. In the summary section we will try to show the connection of these categories to our theoretical model (Nesher, 1980) mentioned in the introduction.

Analysis of the data yielded four main categories (a) to (d) to be detailed below:

a) Linguistic changes and amendments:

- 1) Addition of verbal cues such as "altogether", "more", etc. which may facilitate the finding of the correct operation.
- 2) Replacing some isolated word with others having similar meaning leaving the base story as it is, such as the change of "baskets" to "boxes" or vice-versa.
- 3) Change of anaphoric use to explicit naming, i.e. "his" to "Joseph's".

4) Change in the order of the given sentences.

b) Contextual changes:

- 1) Making the described event more vivid by adding adjectives, adding introduction as a background, adding motives to the actors

in question etc. For example, in problem #1 someone added: "Dan's uncle brough him 4 boxes of crayons from America". Or, "Dan is now confused, can you help him to find out...".

2) Converting a static text into a dynamic one by adding verbs, tenses etc. For example, instead of the sentence "in each of Joseph's baskets..." someone wrote "in each basket Joseph put 3 apples...".

3) Describing the actual actions needed to be performed with the real objects in order to find out what happened. For example, in problem #1 before the question: "How many crayons does Dan have now?", someone wrote: "Dan wanted to show his friend all the crayons that he has in all the boxes, can you tell how many...".

c) Algorithmic advices:

1) Calculating the secondary operation for the child, thus changing the problem into a one-step problem.

2) Asking the child explicitly the secondary question.

3) Reminding the younger child the number given in the beginning, if it is to be used later.

4) Instructing the child to draw the situation described in the text.

d) Non-informative category:

By non-informative we mean acts by the sixth graders from which we could not learn anything. This included:

1) A full solution of the problem by the sixth grader instead of re-phrasing.

2) Mis-understanding of the original problem and changing it to another type of question.

3) Copying the original question without any changes.

TABLE 2 will present the findings according to the above analysis:

TABLE 2

Category	Prob.1.	Prob.2.	Prob.3.	Prob.4.	Total
Linguistic	22	35	15	30	102
Contextual	26	25	19	24	94
Algorithmic	27	18	25	6	76
Non-informative	20	14	23	23	80

* The conclusions will be reported at the meeting.

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SELF-DIRECTED USE OF CONCRETE MODELS IN MIDDLE SCHOOL PROBLEM SOLVING

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RESUME

Cette étude est basée sur une expérience en enseignement faite auprès d'élèves de 7e année. Cette expérience s'intéresse à la capacité de ces élèves à résoudre des problèmes théoriques non-routiniers sur les nombres tout en explorant le rôle d'heuristiques et de modèles concrets et les relations existant entre ceux-ci. Les sujets ont assisté à une série de cours portant sur la théorie des nombres, les heuristiques, les modèles concrets et le micro-ordinateur. Le micro-ordinateur présentait le problème, facilitait le choix d'indices à l'aide d'un menu et gardait en mémoire le travail de l'élève.

Cet article résume quelques aspects de l'étude se rapportant à l'utilisation autonome de modèles concrets durant les phases du projet. Plus spécifiquement, les questions posées sont les suivantes: Y a-t'il un "pattern" observable dans l'utilisation autonome de modèles concrets? Les étudiants qui ont choisi d'utiliser les modèles concrets proposés améliorent-ils leur capacité à résoudre des problèmes? Qu'est-ce qui motive ces élèves à utiliser ou non les modèles concrets? Les élèves dont il est fait mention dans cet article peuvent être répartis comme suit: cinq élèves ont une performance académique supérieure à leur niveau scolaire, cinq ont une performance correspondant à leur niveau et quatre ont une performance inférieure à leur niveau. En premier lieu, il est apparu que le groupe intermédiaire avait tendance à utiliser les modèles concrets proposés plus souvent et que le groupe ayant une performance supérieure à son niveau scolaire avait tendance à les utiliser moins fréquemment. Deuxièmement, il est apparu, tant pour l'ensemble que pour chacun des élèves, qu'une utilisation plus fréquente de modèles concrets correspondait à une plus grande habileté à résoudre des problèmes. Enfin, les élèves montraient peu d'intérêt à utiliser des modèles concrets durant le pré-test et lors de la résolution de problèmes présentés sur ordinateur probablement dû au fait que des indices figuratifs aient été donnés à la fin de la session d'enseignement par l'ordinateur (ou à l'occasion par l'animateur). Les élèves ont prétendu ne pas être sûrs de savoir comment utiliser les modèles, être mêlés par ceux-ci et ne pas éprouver le besoin de s'en servir. Toutefois, il y a eu une forte augmentation de la fréquence d'utilisation de modèles concrets lors du post-test touchant la résolution de problèmes administrés après la phase du projet impliquant l'ordinateur.

INTRODUCTION

This paper reports some aspects of a teaching experiment which focused on seventh graders' performance on nonroutine number theoretic problem solving with a close look at the role of heuristics, concrete models, and the relationships between them. Questions addressed relating to this report are the following: (a) Is there an observable pattern of self-directed use of concrete models? (b) Will students who choose to use provided concrete models improve as problem solvers? and (c) What motivates these students either to use or not use provided concrete models?

Kantowski (1981) astutely pointed out that the tremendous emphasis on computational drill may be counterproductive to the development of the flexibility of thinking essential in problem solving. She asked, "How can this demand for emphasis on basic skills be reconciled with the need for development of problem-solving ability in the limited time available for mathematics instruction?" (p.121) This question has helped put into focus that the problem-solving process for some individuals some of the time is a matter of integrating correct, stable mathematical understandings. For others, perhaps for many individuals most of the time, problem-solving solutions are constructed by slowly and thoughtfully organizing, integrating, differentiating, correcting, and stabilizing mathematical understandings (Lesh, 1982; Lesh & Schultz, 1983). Therefore, problem solving by nature for many means a continued learning of basic mathematical ideas.

Though seventh graders have been taught essentially all elementary school mathematics, they are still refining their understandings of concepts, principles, and algorithms and will continue to do so for some time to come. Regretfully, these children, who are still in transition from the concrete operational to the formal operational stage of cognitive development, are seldom, if ever, provided concrete models to assist in the process of refining their mathematical ideas (Schultz, 1979). Post's (1980) description of concrete models as more abstract than an actual situation, but less

abstract than formal symbols, supports the use of these aids by the problem solver faced with having to process a problem. Because, he reminds us, an important position in mathematics is that if any two systems can be shown to be isomorphic to one another, it becomes possible and desirable to work in the simpler or more available system and transfer all conclusions to the less accessible one. Unfortunately, a common mistaken belief is that manipulatives are more useful in the primary grades than in the middle or secondary grades, and more suited for low ability students than average or above-average students.

PROCEDURE

The setting was a suburban middle-class Atlanta public school (K-7) with a 90% Black population. The seventh graders at this school were in a departmentalized setting in which they attended mathematics class for one hour each day. Five students operating above grade level, five on grade level, and four below grade level are the subjects reported. Groupings were based on results from the California Achievement Test and mathematics placement level at the beginning of the school year. Of the 14, 6 were male and 8 were female; 1 was White and 13 were Black.

The project, which took the entire school year, included all three of the seventh grade classes. Though data for each student were collected, not each student was a subject in the study. In all there were 30 subjects out of 78 seventh graders. The 14 subjects reported here were selected according to the grade level on which they were operating and available data.

Subjects were given a series of lessons on number theory, heuristics, concrete models, and the microcomputer. The microcomputer was used to present the problem, facilitate hint selection through the use of a menu, and to record student work. This paper summarizes some aspects of the study that concern self-directed use of concrete models throughout the phases of the project.

In order to provide a greater feeling for the study, the following schedule is presented. On the first day a General Questionnaire was administered and an orientation to the project was given. This was followed the next day with a Cold Problem-Solving Test (no prior instruction about problem solving) with concrete models available. Then there was a two-week instructional period in which all the basic mathematics content necessary to solve the problems was reviewed, and Polya's four stages of problem solving and heuristics were taught. Provided materials and concrete models were used for all the problems during this teaching intervention. A Pre Problem-Solving Test with concrete models was given immediately after the two-week instructional period, which marked the end of the first phase of the project.

The second phase of the project occurred during the next eight months while the students individually worked nine problems. A problem followed by a menu of hints was presented one at a time on the computer. Concrete models were available at each of two computer stations in the classroom where each of the classes met.

The last phase of the study consisted of a day of administering a Post Problem-Solving Test with concrete models available and a day of administering a Post General Questionnaire and an Attitude Questionnaire.

An important feature not apparent from this description of the design is that individual problems on the Cold Problem-Solving Test, Pre Problem-Solving Test, instructional phase, computer problems and the Post Problem-Solving Test were parallel in structure, e.g., one problem type structure required the student to identify two or three numbers given the product and sum of the factors.

RESULTS AND DISCUSSION

First, there were two ways to search for any observable patterns of self-directed use of concrete models -- through testing data and through computer problem-solving self-report data. The testing data gave the following results. (a) The high group (above grade level) and middle group (on grade level) used concrete models more on the Cold Problem-Solving Test than on the Pre Problem-Solving Test. The middle group dropped from 35% to 15% use of models and the high group dropped from 25% to 15% use of models. The low group (below grade level) actually used concrete models slightly more frequently on the Pre than on the Cold Problem-Solving test showing an increase from 53% to 55% use of the models.

(b) All groups showed a marked increase in use of models from the Pre to the Post Problem-Solving Test. The high group increased from 15% to 67%, the middle group increased from 15% to 83%, and the low group increased from 55% to 71% use of the concrete models. (c) The high group used models as frequently as or less frequently than the other two groups on all three tests. Finally, (d) the highest frequency of use of the models was by the middle group on the Post Problem-Solving Test.

The computer data were written self-reports each subject completed after working each problem presented on the computer. (There are further data collected by the computer on each subject's problem diskette, which are not addressed in this report.) The middle group recorded the highest frequency of 30% use of concrete models, followed by the high group's 12%, and then the low group's 5% use of concrete models.

Second, as a whole, all three groups showed improving problem-solving ability as determined by a weighted score of solution correctness. The percents of solution correctness for the the Cold, Pre, and Post Problem-Solving Tests for the high group were, respectively, 46%-78%-82%; for the middle group they were 40%-49%-62%; and for the low group they were 6%-23%-36%. These

increasing percents correspond to increasing use of the concrete models.

On an individual basis, 12 of the 14 students showed an increased use of concrete models. All but 1 of the 14 improved in problem-solving ability. (This student did show increased use of concrete models.)

Third, two sources of data helped address the question of what motivated these subjects to use or not to use the provided concrete models -- individual interviews with the four students who served as "computer assistants" for each class (one class had two assistants because one became over-involved and had to be replaced) and an Attitude Questionnaire. It was reported that students did not prefer the models on the Cold and Pre Problem-Solving Tests and while working the computer problems because they were not sure how to use them, they became confused by them, or they did not see the need to use them. But while taking the Post Problem-Solving Test, the students realized that all they had was the provided concrete models. They no longer had the computer with a host of hints to call on, some of which were animated illustrations of the problem situation. Students reported that they had to use the models for some problems, that the models were helpful, and some reported that they could not solve any problems without the models.

The testing results suggest that the low level students, who were in over their heads, were attracted toward the concrete models in hopes that they might be more successful at problem solving, which they weren't. The high level students, who used the models the least, apparently simply didn't need them as much as is evidenced by their high scores on the Pre and Post Problem-Solving Tests. The most interesting group was the middle group, which used the models the most both on the test problems and on the computer problems.

The common mistaken notion, mentioned earlier, that concrete models are for use only by low ability students or primary students is not supported by these results. The low and middle groups each improved by 13% on solution correctness between the Pre and Post Problem-Solving Tests, whereas the high group only improved 4%.

These results suggest that if middle school students are shown how the use of concrete models in the problem solving process can help refine mathematical ideas needed in problem solving, and if they are provided models and materials for their use, they will use them and they will improve as problem solvers. Further, children who are of average ability and functioning on grade level are quite receptive and responsive to concrete models in problem solving. More research on this population is warranted.

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INTERPRETATION DE CERTAINES EXPRESSIONS REFERANT A DES
SITUATIONS MATHÉMATIQUES CONCRETES, CHEZ DES ÉLÈVES DU SECOND
CYCLE DU PRIMAIRE

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ABSTRACT

Many studies have contributed to improve our understanding of the processes involved in mathematical written problem solving and of the main difficulties encountered by many children in such a task. Most of the searchers in mathematical problem solving relate children's failures to a misinterpretation of problem's situations. The actual study examines the ways fifth and sixth grades children interpret some expressions representing a few mathematical situations generally included in problem solving tasks. These expressions are presented to thirty children who are invited to transform or to make use of them, in a particular context; the children's results, in a problem solving task, are inferior to the mean results of their peers. Misinterpreted expressions are identified and analysed; children's discourses associated with them are also characterized. Finally, the relations between children's abilities in the interpretation tasks and in the problem solving task are examined.

La résolution de problèmes mathématiques concrets pose, encore présentement, des difficultés sérieuses à une proportion relativement importante d'élèves de différents niveaux scolaires (California Assessment Program, 1979; Clément, 1980; Feeman, 1973). La plupart des chercheurs situent au niveau de l'interprétation des problèmes ou, de façon plus explicite, au niveau de la construction d'une représentation interne de la structure des problèmes, la difficulté majeure des élèves (Krutetskii, 1976; Newman, 1977; Casey, 1978; Hayes, Waterman et Robinson, 1977, 1978).

La construction d'une représentation adéquate d'un problème est influencée à la fois par la structure de l'énoncé (Mayer, 1982; Feeman, 1973; Clément, 1980) et par les schémas de connaissances du sujet (Hinsley, Hayes et Simon, 1977). Les recherches effectuées pour mesurer l'influence de ces facteurs sont très récentes, peu nombreuses et ne procèdent que par une analyse très globale des énoncés des problèmes et des schémas de connaissances des sujets; elles précisent peu les relations entre les compétences linguistiques et les connaissances des sujets d'une part, et leurs processus de résolution de problèmes d'autre part. Elles suggèrent toutefois, diverses explications des difficultés de représentation des problèmes, observées par Krutetskii (1976), chez les élèves faibles et moyens en résolution de problèmes. En effet, la rétention d'informations non pertinentes, de détails superflus et de données numériques isolées caractérisant les élèves faibles pourrait, selon les études mentionnées précédemment, être attribuable à une difficulté de compréhension de certaines expressions ou propositions des énoncés ou encore à l'absence de stratégies pertinentes de lecture des énoncés référant à des situations mathématiques ou encore à l'absence de schémas de connaissances appropriés orientant la lecture significative des énoncés.

OBJECTIFS

L'étude actuelle s'inscrit dans le contexte d'analyse des aspects liés aux difficultés d'interprétation des problèmes. De nature

exploratoire, elle poursuit les objectifs suivants: a. identifier et analyser les interprétations associées à certaines expressions pouvant être incluses dans les énoncés de problèmes chez les élèves faibles en résolution de problèmes mathématiques concrets; b. analyser le discours accompagnant les interprétations produites par ces élèves; c. préciser les relations entre l'interprétation des expressions et le discours qui y est associé et les habiletés en résolution de problèmes mathématiques concrets; d. analyser les difficultés d'interprétation les plus fréquentes.

METHODE

Caractéristiques des sujets

L'échantillon se compose de 20 élèves de 5e année régulière et de 10 élèves de 6e année régulière, provenant d'un milieu socio-économique faible et éprouvant des difficultés dans la résolution de problèmes mathématiques concrets. Une épreuve de résolution de problèmes mathématiques concrets est administrée à 130 élèves de 5e et 6e année régulière; les élèves dont la performance se situe au-dessous de la performance médiane des élèves de leurs classes respectives sont retenus pour l'expérimentation.

Epreuve de résolution de problèmes mathématiques concrets

Quatorze problèmes mathématiques concrets sont présentés aux élèves de 5e et 6e année régulière. Les situations mathématiques représentées dans les divers problèmes sont variées; selon la classification effectuée par Vergnaud (1981), 2 problèmes réfèrent uniquement à des situations additives, 7 problèmes réfèrent uniquement à des situations multiplicatives et 5 problèmes présentent à la fois des situations additives et multiplicatives. Dans chacune de ces catégories de problèmes, les données numériques et les contextes d'application sont variés; les quantités ne dépassent toutefois pas 10 000 et les contextes principaux sont l'alimentation (4 problèmes), le commerce (5 problèmes), les sports (4 problèmes) et la famille (1 problème).

Cette épreuve est administrée aux élèves des différentes classes, en deux parties. Dans une première partie, les huit premiers problèmes sont présentés; les autres problèmes sont présentés, dans la seconde partie. Deux séances d'une heure chacune sont requises pour l'administration de l'épreuve. Enfin, deux modes de correction de cette épreuve sont appliqués: pour la sélection des sujets, un point est alloué à chaque problème réussi (cote maximale: 14 points); pour l'analyse des performances en résolution de problèmes, des sujets qui participent à l'expérimentation, un point est attribué à chacune des situations mathématiques correctement interprétées (cote maximale: 28 points).

Epreuve de traitement d'expressions référant à des situations mathématiques concrètes

A. Description

L'épreuve de traitement d'expressions référant à des situations mathématiques concrètes est élaborée à la suite d'une analyse des processus de résolution de problèmes d'un petit groupe de sujets présentant des difficultés en ce domaine. Ces sujets obtiennent des résultats comparables à ceux des sujets retenus pour l'expérimentation, à l'épreuve de résolution de problèmes décrite précédemment. Les échecs les plus importants, relevés au cours de cette analyse, proviennent d'une interprétation inadéquate des expressions impliquant des relations entre des mesures et des expressions référant à des situations multiplicatives de formes "x par y", "x à chacun des y". Ces erreurs sont reconnues aussi comme très fréquentes par d'autres chercheurs en ce domaine (Mayer, 1982, Newman, 1977).

L'épreuve ainsi construite comporte 9 expressions; 2 de ces expressions réfèrent à des situations multiplicatives, 4 indiquent une relation entre des mesures, 2 font référence à des actions et une dernière ne présente qu'une mesure d'une collection. Ces expressions sont présentées selon la séquence suivante: 1. 125 billes; 2. partager également; 3. 450 paquets de gomme pour 2 familles de 5 enfants chacune; 4. 450 paquets de gomme à chacun des enfants,

2 familles de 5 enfants chacune; 5. comparer, économiser; 6. 5 fois plus âgé que; 7. 8 heures de moins que; 8. 30 centimètres de plus que; 9. 15 fois moins long que.

B. Mode de passation

Les sujets sont répartis dans 6 sous-groupes de 5 sujets chacun; trois expérimentateurs sont requis pour la présentation des exercices et chacun d'eux se voit assigné 2 sous-groupes de sujets. Trois séances de 50 minutes chacune sont nécessaires à l'administration de l'épreuve. Dans un premier temps, chacun des sujets d'un sous-groupe est invité à modifier une situation, à traiter un événement ou à appliquer une transformation, puis à inscrire le résultat obtenu sur une feuille. Dans un second temps, deux sujets recueillent toutes les réponses et procèdent à une classification de ces réponses, en regroupant celles qui leur semblent résulter d'une interprétation similaire de l'expression initiale. Dans un dernier temps, tous les sujets expliquent leurs réponses respectives et suite à cette explication, les sujets ayant effectué une première classification peuvent, s'ils le jugent opportun, modifier cette classification. Tous les comportements des sujets sont enregistrés sur cassettes.

C. Analyse des comportements

L'interprétation de chacune des expressions, le discours relié à chacune d'elles font l'objet d'une évaluation. L'évaluation est appliquée aux comportements reliés aux différentes étapes du traitement des expressions. Elle tient compte ainsi des précisions et des modifications apportées par les sujets au cours de ces différentes étapes. Parmi les interprétations produites par les sujets, certaines sont mathématiques tandis que d'autres sont de nature linguistique; par exemple l'expression "comparer, économiser" est traitée par un sujet de la façon suivante "il faut regarder deux choses pour comparer et choisir le moins cher pour économiser" et par un autre sujet, de cette façon "je vois 2 lbs de beurre, une est 98¢ et l'autre 1,07\$; j'économise 9¢". Dans l'analyse, ces deux types d'interprétation sont distingués. De plus, les interprétations mathématiques adéquates sont distinguées des interprétations

inadéquates. Enfin, pour chacun des sujets, la fréquence des différentes catégories d'interprétation est notée (cote maximale: 11 points).

L'examen du discours des sujets fait enfin l'objet d'une attention particulière. Ainsi, pour chacun des sujets et pour chacune des expressions traitées, un point est alloué pour l'emploi judicieux de chacune des composantes suivantes: pronoms personnels, pronoms et adjectifs possessifs, noms et verbes, syntagmes prépositionnels; un dernier point est aussi accordé pour la structuration adéquate des propositions (cote maximale pour les 9 situations: 45 points).

RESULTATS

Habiletés de traitement d'expressions "mathématiques" et habiletés de résolution de problèmes mathématiques

Dans le but d'analyser les relations entre les habiletés de traitement d'expressions référant à des situations mathématiques et les habiletés de résolution de problèmes mathématiques, les performances des sujets aux épreuves mesurant ces habiletés sont ordonnées selon différents niveaux. Cinq niveaux de performances en résolution de problèmes (épreuve A) sont établis: 0-5, 5-10, 10-15, 15-20, 20 et plus; 4 niveaux d'interprétation des expressions (épreuve B) sont distingués: 0-4, 4-7, 7-10, 10 et plus; enfin, 5 niveaux de discours (épreuve C) sont distingués: 15-20, 20-25, 25-30, 30-35, 35 et plus. Ces niveaux sont tous déterminés à la suite d'une analyse de la distribution des cotes aux différentes épreuves.

Les habiletés de traitement d'expressions référant à des situations mathématiques concrètes et les habiletés de résolution de problèmes mathématiques concrets, chez les enfants éprouvant des difficultés en résolution de problèmes semblent liées. En effet, les sujets qui ne réussissent à interpréter correctement que certaines expressions (épreuve B: niveau I: 0-4) ne peuvent résoudre correctement que quelques problèmes (épreuve A: niveaux I, II: cotes inférieures à 10); par ailleurs, la majorité des sujets qui interprètent

correctement plus de la moitié des expressions réussissent plus que les $\frac{2}{3}$ des problèmes mathématiques. La qualité du discours accompagnant l'interprétation des expressions est aussi liée à l'habileté de résoudre des problèmes mathématiques; la majorité des sujets dont le discours est acceptable (épreuve C: niveaux III, IV et V: cotes 25 et plus) réussissent plus que les $\frac{2}{3}$ des problèmes mathématiques.

Enfin, l'interprétation des expressions et la qualité du discours accompagnant cette interprétation sont aussi des aspects associés dans le traitement des expressions; ainsi, un discours pauvre se retrouve généralement chez les sujets qui ne réussissent à interpréter correctement moins que la moitié des expressions, en revanche, les sujets qui produisent un discours acceptable (épreuve C: niveaux III, IV et V) sont aussi ceux qui peuvent interpréter adéquatement la majorité des expressions (épreuve B: cotes supérieures à 7).

Afin de préciser les relations entre les habiletés de traitement d'expression référant à des situations de type additif et les habiletés de résolution de problèmes de ces deux types, les résultats moyens des sous-groupes de sujets dans le traitement de ces deux types d'expressions et dans la résolution de problèmes incluant des situations multiplicatives et additives sont mis en relation. La proportion d'expressions référant à des situations multiplicatives correctement interprétées par les sujets des divers sous-groupes est supérieure à celle des problèmes de type multiplicatif réussis par ces mêmes sujets. Par ailleurs, à l'exception d'un sous-groupe (sous-groupe 5A1), les sous-groupes dont la performance dans l'interprétation des expressions est égale ou supérieure à 60%, sont aussi ceux qui montrent une meilleure performance en résolution de problèmes. De plus, les sous-groupes qui présentent une meilleure qualité du discours sont aussi ceux qui obtiennent les meilleures performances dans les problèmes mathématiques de type multiplicatif. Ainsi, le sous-groupe 6A dont le discours présente les meilleures qualités est aussi le sous-groupe dont la performance en résolution de problèmes est la plus satisfaisante, avec une réussite de 75% des problèmes.

Les relations entre l'interprétation juste des expressions de type relationnel, la qualité du discours accompagnant ces interprétations et la réussite des problèmes incluant ce type de relations, apparaissent difficile à préciser. En effet, le sous-groupe qui obtient la plus faible performance moyenne dans l'interprétation des expressions, tout en utilisant un discours comparable à celui des autres sous-groupes (sous-groupe 6A), est celui qui réussit le plus de problèmes incluant des situations relationnelles. De plus, le sous-groupe qui obtient la meilleure performance dans l'interprétation des expressions "relationnelles" et utilise un discours supérieur à celui des autres (sous-groupe 5A1), ne réussit que 20% des problèmes de ce type; ce taux de réussite est le plus faible.

L'interprétation inadéquate des expressions référant à des situations mathématiques et la pauvreté du discours accompagnant ces interprétations inadéquates semblent liées aux difficultés de résolution de problèmes mathématiques concrets. Un examen des commentaires des sujets au cours de l'expérimentation indique de plus que les sujets qui interprètent incorrectement certaines expressions proposées et qui produisent un discours confus sont conscients de leurs erreurs; ces sujets ont aussi tendance à modifier leurs premières interprétations, à la suite des présentations de celles de leurs camarades, bien qu'ils ne soient pas en mesure de justifier adéquatement leurs seconds choix. Enfin, l'analyse des interprétations inadéquates effectuées par plusieurs sujets montre qu'un bon nombre de ces sujets modifient certaines expressions soit en y substituant certains éléments, soit en faisant abstraction de certaines relations. Ces modifications sont particulièrement remarquables dans les situations de type multiplicatif et de type relationnel.

Les résultats de cette étude permettent de mieux comprendre les difficultés d'interprétation des problèmes mathématiques relevées par Krutetskii (1976), Feeman (1973) et Clément (1980), chez les élèves faibles. En effet, la sélection d'informations non pertinentes et de données numériques isolées et l'ignorance des relations entre les données numériques caractérisant la démarche d'interprétation des

problèmes des élèves faibles pourraient traduire une difficulté de compréhension de certaines relations ou de certaines expressions référant à des situations mathématiques, amenant alors ces élèves soit à ignorer ces informations, soit à les modifier pour qu'elles deviennent significatives, soit enfin, à leur préférer d'autres informations, par exemple des éléments contextuels et des données numériques et à élaborer, à partir de ces informations, un autre problème. Il apparaît alors essentiel d'examiner, dans une étude systématique auprès d'élèves faibles, moyens et forts, ces différents facteurs explicatifs.

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K. TEACHER TRAINING / FORMATION DES ENSEIGNANTS

EXPOSÉS RELIÉS ÀS RELIÉS IN PRACTICE: THE CASES OF FRED AND JANICE

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RESUME

Le présent exposé poursuit la présentation faite au PME-NA 1988 dans laquelle nous avions parlé de la recherche conduite dans le cadre d'un projet subventionné par la NSF, et portant sur les croyances des enseignants. À cette occasion les croyances de quatre enseignants avaient été décrites. Dans le présent exposé nous faisons un bref rappel des techniques d'interview, lesquelles avaient principalement utilisé des situations hypothétiques, puis nous décrivons l'enseignement fait en classe par deux des quatre enseignants en question.

Des copies des rapports analysant leurs croyances, telles que révélées pendant les interviews initiales furent données aux deux enseignants Fred et Janice. La réaction de Fred fut longue du fait qu'il profita de ce moment pour discuter de ses neuf premières semaines d'enseignement. Un certain conflit, entre les vues qu'il avait adoptées et sa pratique dans la classe, spécialement en ce qui touche la résolution de problèmes, devint évident. La réaction de Janice fut brève puisque selon elle, le rapport avait "capté" assez bien ses croyances.

En novembre l'un observa les classes de Fred pendant neuf jours consécutifs et ensuite ses élèves et lui furent interviewés. L'analyse des croyances auxquelles il tient et des croyances telles qu'elles apparaissent dans sa pratique sont comparées, particulièrement en ce qui a trait aux croyances reliées à son utilisation des problèmes dans la classe.

Les classes de Janice furent observées vers la fin de l'année pendant huit jours repartis sur une période de deux mois. Janice, ses étudiants et l'une de ses collègues furent interviewés. Même si cette analyse n'est pas complétée, il semble clair que l'orientation socio-psychologique de Janice envers l'enseignement, orientation qui était évidente durant les interviews initiales, était aussi présente dans la classe. Les implications sont discutées.

ESPOUSED BELIEFS AND BELIEFS IN PRACTICE: THE CASES OF FRED AND JANICE

by
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At the 1982 PME-NA meeting in Athens, Georgia, a presentation was made on a NSF supported research project being conducted by Stephen Brown and Douglas Smith (SUNY at Buffalo), Catherine Brown (University of Georgia) and myself. In that paper we described the interview process which was used with four teachers as we investigated each of the informant's beliefs about mathematics and the teaching and learning of mathematics. This report represents an update at least for some of the Project's activities. In particular two of the individuals (Fred and Janice) who were briefly described in the 1982 PME-NA paper were followed into the classroom as their classes were observed and interviews were conducted with them and their students. A brief summary of the interview process will be given followed by an analysis of the classroom observations and subsequent interviews.

THE INTERVIEW PROCESS

The two informants, Fred and Janice, were completing their master's degree programs at the time that the initial interviews took place. Fred's master's degree program was completed prior to his first year of teaching. Janice started her master's degree program during her first year of teaching and then spent a year at the University of Georgia completing her program prior to her second year of teaching.

For both individuals the initial interviews consisted of seven sessions which were conducted during the winter and spring quarters of 1982. The interviews focused on their reactions to a number of primarily hypothetical situations called episodes. The episodes were created for the purpose of stimulating conversations about the nature of mathematics and the teaching and learning of mathematics. The position was taken that while teachers are not likely to have well-articulated theories about the teaching and learning of mathematics, they will

have implicit theories which may be expressed when provided with a variety of different stimuli. The episodes varied with regard to open endedness (e.g., some required reaction to stated positions and some gave no indication of the range of possible answers), to "voice" that was expected in response (e.g., sometimes the informants assumed that (s)he was responding as herself/himself, sometimes as someone else), and to a sense of real worldliness (e.g., some were quite literal and others required a leap of imagination).

After four interviews, all transcribed, the informants were asked to identify those statements in the transcripts which they felt were particularly important expressions of their beliefs. These statements were then subsequently grouped and the interviewee created a title and brief descriptor for each set of statements. This "clustering" process is an adaptation of the constant comparison method used in qualitative data analysis. The interview process and, in particular, the clustering process provided the basis for reports on the beliefs espoused by Fred and Janice. The respective reports were subsequently shown to both individuals for their reactions. Subsequently, the investigator followed both of the teachers into their classrooms for observations and interviews. The results of these activities are described below.

FRED: CONFLICT AND ENCULTURATION

Fred's beliefs as expressed during the interviews fall basically into four "categories": beliefs about himself, about mathematics, about teaching, and about learning. An overview of this analysis was presented at the last PME meeting. Briefly, he views himself as one who aspires to be wise. "The ideal teacher reacts to unruly students in wise ways." "I wish I had the wisdom of Solomon." He wouldn't mind being Polya because "he was a master of mathematics and teaching." If he could be a famous person (Evidently, Fred did not consider Polya to be famous!), he would pick Einstein. "Who wouldn't want to be really bright." Fred expressed a strong belief that "mathematics is essentially problem solving." "Problem solving is the essence of

mathematics and heuristics is the essence of problem solving." For Fred mathematics is fun. The intersection of fun and mathematics lies in solving recreational problems. Fred believes students learn best when they are motivated and excited about mathematics and when they find learning "fun". He complained that teachers and texts alike require too much drill and practice.

After nine weeks of teaching Fred was asked to react to the report. The investigator intended to receive feedback on what aspects of the report "accurately" described his beliefs and which parts did not. Surprisingly enough, Fred's reaction was much less concerned with the report's "accuracy" (Basically he thought it described his beliefs quite well.) and much more on using the report as a forum for discussing his first nine weeks of teaching. His reaction was lengthy. In short he realized a certain conflict between his beliefs espoused prior to teaching and his actual teaching performance. He realized the difficulty of trying to make problem solving the focal point of his instruction. At one point he stated,

I'm less and less confident in my own conception of what constitutes a problem-solving episode in a high school classroom.

In describing his attempt to use the flagpole problem (Find the height of the flagpole using similar triangles.), he summarized his reaction by saying, "The vast majority of the students cared less." At another point he stated,

So far I am finding that for all my recreational math problems that I pose to them and for all my efforts of explanation and good pedagogical techniques, I still don't know how to motivate students.

Having taught now for nine weeks it's very clear how much I am able to integrate this problem-solving attitude into my teaching. The answer is not very much -- to my own consternation.

In short, Fred realized a hiatus between his beliefs about problem solving and what he was able to actualize in practice. That hiatus caused him some pause about his own teaching.

In November Fred's classes were observed on nine consecutive days and interviews were conducted with him and his students. The conflict that had been realized earlier became increasingly evident. When teaching, he seemed to interpret problem solving as a technique of presenting interesting problems for the purpose of capturing the interest of students. He hoped that the basic content would then be more readily "acceptable". The following statements are representative of his views on the role problems play in his classroom teaching.

My intention was not so much a problem-solving situation but as sort of an interest creator.

I intentionally used that (problem) during the first period class just as some thing fun for us to look at before we get down to brass tacks.

Observations of Fred's classes revealed little emphasis on heuristics. Virtually no mention, implicit or explicit, of heuristics occurred when problems were discussed. Fred's "inservice view" of the importance of heuristics was considerably different than his "preservice view".

Fred's teaching style was viewed by advanced students as a breath of fresh air when compared with their previous mathematical studies. "He doesn't let the class get boring -- like him just sitting there talking and nobody is paying any attention." "He keeps it lively by putting in little games and stuff to do with mathematics." But the general mathematics students viewed his style with contempt. They interpreted his actions as wanting to make a joke out of teaching mathematics -- at a time when many of them were failing. One student claimed that the class "never gets down to serious work". Serious work, at least for one student, constitutes "adding, subtracting, multiplying, and dividing fractions and decimals." Several students complained about such "silly" things as rolling dice. (The lesson was on probability and insurance.) Another student commented, "If he'd do a better job of teaching and worried about us getting it, not that many would have failed."

It seemed clear that the enculturation process of being a first year teacher was causing Fred to reevaluate his beliefs about the teaching and learning of mathematics and perhaps about himself. His views on problem solving and on the role of heuristics is but one example of a chasm between espoused beliefs (as interpreted by the investigator) and those beliefs evident in practice.

JANICE: PATIENCE AND FRUSTRATION

In a similar vein, a report was written about Janice's beliefs as expressed during the initial interview phase. She, too, proclaimed the importance of problem solving and applications. However, when pressed for specifics, she could not identify specific problems or applications that she had used or hoped to use in her second year of teaching. She indicated this was an area where she needed help.

It seemed that Janice's conceptual systems were more grounded in social/psychological considerations than in subject matter oriented ones. During the clustering process she identified the following statements as being particularly important to her.

I would like students to feel better about themselves as math people.

I would like to eliminate all negative interaction from other people.

An ideal teacher is a personable person -- just someone that the students enjoy being around.

In general Janice saw herself as a person who wants to promote a caring and supportive classroom atmosphere.

The report on Janice's beliefs was also given to her for her reaction. Her comments were relatively brief. She expressed few reservations and indicated that the report did an excellent job of describing what she believed.

Near the conclusion of her second year of teaching her classes were observed for eight days over a two month span. Interviews were

conducted with her, one of her peer teachers, and several of her students. A final interview was conducted with Janice after the close of the school year in an effort to provide more detailed information about her beliefs and to probe origins of those beliefs.

The data are now being analyzed. Preliminary indications suggest that the social/psychological orientation which seemed to be present during the initial interviews was also present during her teaching and subsequent interviews. She exudes patience yet expresses considerable frustration when students fail to perform to her satisfaction. When students' performances disappoint her, she reasons that the student is bright or capable but that motivation or various social factors impede better performance. With some students she assumes much of the responsibility for their lack of progress. She does so in a somewhat self-debasing way. The following comment seems typical.

I said sometimes I feel like I've failed because I'm not the spectacular teacher and I'm not maybe the teacher for you because you are pretty bright and he just shrugged it off.

Too, there seems to be a certain "why are you doing this to me" tone in her voice when students misbehave or are obviously not paying attention. The contexts of such remarks require further analysis but there seems to be a hint that this sort of reaction to students' inattentiveness or disruptions has its roots in her interactions with her parents when she misbehaved as a child.

She speaks of "being put to shame" by some of the high PSAT scores of the students -- as if she needed to apologize for her scores. She talks, as did Fred, about how the students "like mechanical stuff." "If here is a procedure and here is a definition and this is how they work and this is what we do with it -- they just love that. It seems like they love to do busy work that is very procedural." Hence, her attempts to "liven up" presentations by introducing a problem or by showing different algorithms were not rewarded, generally speaking, by student enthusiasm.

Janice made extensive efforts to bring more to the classroom than just textbook mathematics. But her efforts were often thwarted either by student lack of interest or by her own limitations, e.g., lack of planning time or a lack of knowledge.

A FINAL COMMENT

At the 1982 PME-NA meeting there were several alternative directions identified for the project. In our second year part of the project staff opted, essentially, for the first alternative, viz., to study espoused beliefs and beliefs in practice. Based on the analysis of Fred and a preliminary analysis of Janice there seems to be many implications for further research and for teacher education. It appears we can not assume (Do we?) that the conceptual models presented in our teacher education programs are the same models the students take with them into the classroom. That is, the intended teacher education curriculum may not be the same as the learned curriculum. Fred espoused the importance of problem solving. There was an inherent and perhaps erroneous assumption on the part of the investigator that Fred's use of the term problem solving was nearly synonymous with his own. But Fred's utilization of problem solving (or at least of problems) indicates that either his thinking underwent a dramatic shift when he started teaching or that the investigator failed to comprehend fully what Fred was saying about problem solving during the initial interview phase. Implications exist for both the researcher and the teacher educator.

The teacher's world in the classroom is a complex one. We struggle to provide teachers with conceptual models which enable them to operate in more than a common sense mode. But teachers have pressures and considerable noise (figuratively and literally) which divert their attention. Students are not always appreciative of efforts to do more than textbook mathematics. The strain of finding problems and applications is another factor which mitigates against more creative teaching styles. Yet, Fred and Janice persisted in doing more than the minimal. It is not clear, however, what directions their efforts will lead them, for the seeds of dissatisfaction have already been sown.

PROGRAMMATION EN LOGO ET FORMATION DES ENSEIGNANTS

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This article reports some observations done within a graduate course where education students introduced themselves to LOGO programming and analysed their own problem solving protocols. It is shown that these activities were an occasion for very intense and personal contact with logico-mathematical thinking and problem solving.

LOGO constitue un environnement informatique cherchant à rendre accessible l'activité de programmer un ordinateur et à permettre d'en exploiter le potentiel éducatif (Papert 1981). Cette recherche se situe dans le contexte de travaux visant à utiliser cet environnement dans le domaine de la formation des enseignants (Austin 1976 ; Du Boulay 1977). Nous nous intéressons à étudier en quoi de telles activités de programmation peuvent amener quelqu'un à s'impliquer profondément dans une démarche d'apprentissage et de ce fait constituer une situation privilégiée pour l'aider à prendre contact avec son propre fonctionnement en tant qu'apprenant, et en particulier sa façon d'aborder les connaissances logico-mathématiques et la résolution de problèmes. Il ne s'agit pas d'une formation en didactique des mathématiques, au sens strict, mais du développement d'un type de compétence qui à la fois soutient et dépasse le domaine des mathématiques. On peut la décrire simplement comme étant une certaine habileté à exprimer quelque chose de personnel à travers un langage formel et une certaine confiance en sa capacité de le faire.

Nous rapportons ici des observations effectuées dans le cadre d'un cours gradué en psychopédagogie intitulé "Practicum en analyse de processus d'apprentissage". A l'hiver 1983, un groupe de 12 étudiants se sont introduits à LOGO à raison de deux heures de travail à l'ordinateur par semaine pendant l'équivalent de dix semaines. Ils ont documenté et analysé leur démarche à l'aide d'un journal de bord incluant les traces sur papier d'imprimante de tout ce qu'ils ont tapé sur le clavier de l'ordinateur. De plus ils ont eu l'occasion de présenter leur travail et d'échanger sur leur expérience lors de trois séminaires répartis dans le semestre.

Chaque participant a été introduit à LOGO à travers la géométrie de tortue. C'est le nom donné à un petit triangle apparaissant sur l'écran graphique et que l'on peut faire déplacer ou pivoter à l'aide d'instructions telles AVANCE 100 ou DROITE 90. Cette tortue contient

un crayon qui peut être à l'état "levé" ou "baissé", prendre différentes couleurs, et même être remplacé par une gomme à effacer. Programmer en LOGO consiste à donner un nom à une série d'instructions qui, dans le contexte de la tortue, peuvent générer toutes sortes de figures. Une caractéristique fondamentale de la programmation en LOGO est le fait qu'une fois qu'un nouveau mot (une nouvelle procédure) est défini, il acquiert le même statut que ceux du vocabulaire primitif. Il peut être utilisé pour en définir de nouveaux. On obtient alors une organisation hiérarchique de procédures et sous-procédures qui encourage naturellement des stratégies de décomposition d'un problème en sous-problèmes.

Le scénario d'enseignement que nous avons utilisé a consisté à chercher à s'ajuster aux intérêts et initiatives de chacun. Aussi au départ nous n'avons présenté qu'un minimum de commandes permettant de faire fonctionner la tortue, de répéter une série d'instructions, de construire, d'éditer et de sauvegarder des procédures. Les notions de variable, de récursion et d'embranchement, de même que les opérations arithmétiques et logiques, les listes et autres possibilités de LOGO n'ont été présentées que lorsque requises par un projet. Tous les participants ont commencé par réaliser au moins un projet avec la tortue. Par la suite, plusieurs se sont intéressés à la génération de phrases ou à la programmation de diverses formes de dialogues et de jeux. Dans la seconde moitié du semestre, certains ont introduit d'autres personnes à LOGO, dont de jeunes enfants.

UNE EXPERIENCE DE CONTACT AVEC LA REALITE

Il n'est pas facile d'essayer de rapporter ce mélange de fascination et d'irritation qui a semblé caractériser leur expérience. Voici quelques remarques enregistrées lors du premier séminaire :

"Je trouve ça très captivant. Je trouve que les deux heures passent très très vite."

"Je trouve ça très difficile, très absorbant, mais emballant. C'est très exigeant. Tu ne peux pas te permettre de distractions parce que tout de suite tu fais des erreurs. Ça demande énormément de concentration."

"Je m'attendais tellement à rater avec cette machine qu'à chaque fois que ça réussit je suis surprise. J'y passe beaucoup de temps. C'est toujours plus ou moins présent à mon esprit. C'est un espace de passion qui vous mine. (rires)"

"La tortue m'a rappelé que je ne suis pas bonne en mathématique. Quand je bloquais en math, j'avais de la difficulté à voir le problème sous une autre facette. J'essayais d'obtenir la solution de quelqu'un ou je passais à autre chose. Ici je fais la même chose. (...) Malgré tout je trouve ça intéressant."

"Ca me déçoit de prendre tant de temps juste pour faire cette maison. Je voulais travailler en français. Mais c'est plus fort que moi, il faut que je la finisse."

"Je trouve qu'il y a beaucoup de choses à faire. Ca ne va pas assez vite. Je ne veux pas passer toute la session sur mon bateau."

Apprendre à programmer exige constamment de la rigueur et de la précision. Il y a d'innombrables occasions de se tromper ne serait-ce qu'en faisant une erreur de frappe ou en oubliant de laisser un espace entre deux mots. Mais à chaque instruction, on obtient immédiatement un feedback. S'il y a erreur de syntaxe, LOGO donne un message qui indique très précisément où elle se trouve et de quel type elle est. On entre donc dans une situation d'interaction qui peut devenir très intense. On veut obtenir un certain résultat. On essaie. Cela réussit partiellement ou pas du tout. On essaie de comprendre. On recommence en modifiant quelque chose... On devient facilement captivé par ce processus, par le besoin de se rendre au bout du projet. Si on ne réussit pas, on se sent extrêmement frustré. Si on réussit, on a de la misère à ne pas crier de joie. Mais en même temps, comme il y a continuellement beaucoup de petits détails à maîtriser, on a l'impression que ça ne va pas très vite. On croyait construire cette maison en une séance. Cela nous en prend six ou huit.

Il y a là une sorte de contact très direct avec la réalité d'un monde purement logique qui agit comme un miroir et nous renvoie une image exacte de ce qu'on exprime. Au début, plusieurs avaient plus ou moins

l'impression que l'ordinateur "comprenait" ce qu'ils lui tapaient. Ils n'arrivaient pas à voir qu'il est possible de taper CARRE et que la tortue dessine une sorte d'escalier. Ce n'est que graduellement qu'ils ont réalisé qu'ils avaient affaire à une machine à traiter des symboles qui interprète ses instructions de façon mécanique à partir des règles qu'elle possède. D'où une remarque comme :

"Je réalise que la machine est bête. Il faut tout lui dire. Elle ne sait rien."

COLORIER UN RECTANGLE

La séquence suivante provient d'un projet qui consistait à construire une sorte de jeu de labyrinthe avec la tortue. Pour ce faire, il fallait placer sur l'écran diverses formes pleines qui allaient servir d'obstacles. Comme la forme rectangulaire revenait souvent dans son plan, l'étudiante avait décidé de construire une sous-procédure permettant de colorier un rectangle. Voici sa démarche.

- 1) Elle avait précédemment défini une procédure à deux variables permettant de tracer un rectangle en spécifiant ses dimensions, soit :

```
POUR RECTANGLE :LAR :LON  
REPETE 2 [AV :LAR GA 90 AV :LON GA 90]  
FIN
```

- 2) Elle rapporte dans son journal avoir simulé sur papier, avant la séance, une procédure permettant de colorier un rectangle de 50 par 25. Arrivée à l'ordinateur, elle commence par taper :

RECTANGLE 50 25

Puis fait :

```
REPETE 25 [GA 90 AV 25 DR 90 AV 1 DR 90 AV 25 GA 90 AV 1]
```

Elle obtient effectivement un rectangle plein.

- 3) Maintenant elle définit sa procédure générale pour colorier un rectangle :

```
POUR COULEUREC :LAR :LON  
REPETE :LAR [GA 90 AV :LAR DR 90 AV :LON DR 90 AV :LAR  
GA 90 AV :LON]  
FIN
```

Puis elle fait :

RECTANGLE 50 25 puis COULEUREC 25 1

Elle obtient son résultat et semble satisfaite. Elle passe ensuite à la définition d'une procédure pour colorier un triangle, qu'elle réussit.

- 4) A la séance suivante, elle veut colorier un carré et tape :

CARRE 25 puis COULEUREC 25 25

Elle obtient quelque chose d'aberrant. Elle examine sa procédure puis essaie :

COULEUREC 50 1

Ensuite elle fait trois autres essais, soit :

CARRE 50 avec COULEUREC 50 1

RECTANGLE 100 25 avec COULEUREC 25 100

RECTANGLE 100 25 avec COULEUREC 25 1

Comme aucun ne réussit, elle passe à autre chose. Plus tard elle écrit dans son journal avoir été "surprise" et "déroutée" de constater que COULEUREC "ne fonctionnait plus".

- 5) A la fin de la séance, elle définit la superprocédure LABYRINTHE qui doit placer ses différentes figures aux bons endroits sur l'écran. Elle y inclut plusieurs utilisations de COULEUREC. Elle essaie le tout et, constatant que cela ne fonctionne toujours pas, décide de supprimer la couleur. Elle note cependant dans son journal qu'en faisant cela, elle a "mis le doigt sur le bug. COULEUREC respecte toujours la largeur mais pour certains rectangles, elle modifie la longueur."
- 6) A la séance suivante, elle reprend le problème et travaille à modifier l'utilisation de la variable :LON. Elle réalise que celle-ci doit servir à spécifier le nombre de fois qu'il faut répéter la séquence d'instructions et comme à chaque coup la tortue trace deux traits, elle doit la répéter :LON / 2 fois, ce qui donne :

POUR COULEUREC :LAR :LON

REPETE :LON / 2 [GA 90 AV :LAR DR 90 AV 1 DR 90 AV :LAR

GA 90 AV 1]

FIN

Cet épisode est caractéristique d'un type de confusion que chaque participant a connu à maintes reprises et qui provient de la difficulté à intégrer de façon cohérente divers éléments de connaissance. Ici on voit au départ qu'elle est capable d'utiliser la notion de variable, puisqu'elle a construit une procédure RECTANGLE avec deux variables (#1). De plus elle peut simuler sur papier une série d'instruc-

tions, ce qu'elle a fait avant la séance pour trouver comment colorier un rectangle de 50 par 25 (#2). Mais lorsqu'elle veut généraliser le tout en une procédure pouvant colorier n'importe quel rectangle, elle introduit une certaine confusion (#3). Elle utilise les mêmes noms pour les variables que ceux de RECTANGLE, mais ici :LAR correspond à la variable :LON de RECTANGLE et :LON doit nécessairement prendre la valeur 1. Elle exécute sa procédure une seule fois, avec les valeurs appropriées correspondant cependant à son exemple de départ. Mais lorsqu'elle veut réutiliser COULEUREC avec d'autres valeurs, à la séance suivante, elle a complètement perdu la trace de cet espèce de glissement qui s'est produit dans la définition des variables. Elle prend pour acquis que :LON veut dire longueur et :LAR, largeur. Aussi pour obtenir un carré, elle fait COULEUREC 25 25 et se trouve complètement désorientée par le résultat qu'elle obtient (#4). On a ici un bel exemple de "dissonance" entre ce qu'on a défini et ce qu'on croit avoir défini.

Sa réaction à cette situation en est une de surprise, de déception et d'impuissance. Elle fait quatre autres essais infructueux, en revenant au rectangle mais sans reproduire l'exemple qui avait fonctionné précédemment. Puis elle passe à autre chose. Mais à la fin de la séance elle y revient comme si par magie sa procédure allait soudainement fonctionner correctement (#5). Elle a d'ailleurs écrit dans son journal :

"J'essaie souvent la même chose voulant me convaincre que la machine se réveillera et répondra à mon désir."

Mais en faisant cela, elle a quand même réalisé que la largeur était correcte et qu'il y avait quelque chose de bizarre relié à la variable :LON, ce qui lui a permis de résoudre son problème, à la séance suivante (#6).

Nous voyons dans cet épisode un exemple courant d'une sorte de résistance à affronter directement un problème. Cette étudiante ne semble pas confiante dans sa capacité de comprendre exactement comment LOGO

fonctionne. Pourtant elle était capable de simuler sa procédure, instruction par instruction, et si elle l'avait fait elle aurait vu tout de suite ce qui ne fonctionnait pas. Mais en même temps, on voit que l'aspect interactif de la relation avec l'ordinateur, le fait qu'on puisse répéter indéfiniment l'exécution d'une procédure, lui a permis de garder le contact avec son problème, même si c'était de façon plutôt globale, et éventuellement de le résoudre.

Pour pouvoir établir une relation "saine" avec les mathématiques, il faut se développer un certain sens de ce que c'est que de formuler et de manipuler des expressions symboliques, et une certaine confiance en sa capacité de le faire. Nos observations ne nous permettent pas de parler de progrès spectaculaires ou de changements radicaux d'attitudes. Mais nous croyons que ces activités de programmation ont permis à chaque participant d'aborder la pensée logico-mathématique et la résolution de problèmes de façon intense et personnelle. Et nous voyons là tout un potentiel à exploiter.

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QUELQUES PROBLEMES LIES A LA PERCEPTION
QU'ONT LES MAITRES
DES MATHEMATIQUES ET DE L'APPRENTISSAGE

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ABSTRACT

Four problems, tied to the nature of mathematics, to the philosophy of mathematics, to epistemology and to learning theories, have influenced mathematics education for years, and still do. These problems are the insistence put on symbolism and notation, the great influence of formalism, the heavy presence of behaviorist learning theories and the fact that many teachers are focusing on their pupil's answers rather than on their reasoning.

These problems are perpetuated through many ways; the two main ones are the programs (curriculum) among whose sources are the very sources of the mentioned problems and the teachers themselves, whose preparation is less than adequate, preventing them from coping with the situation. The fact that this preparation is lacking a minimum of integration, its components - psychology, pedagogy and mathematics - being set apart, certainly does not help.

This paper propose that the only way to remedy the situation is to explore new approaches for teacher training, one of these, which is currently experimented, being shortly described.

QUATRE PROBLEMES MAJEURS EN DIDACTIQUE DES MATHÉMATIQUES.

Quatre problèmes ont affecté et affectent toujours de façon chronique la didactique des mathématiques, problèmes d'autant plus ardues qu'ils touchent des questions fondamentales reliées à la nature des mathématiques elles-mêmes, aux philosophies qui sous-tendent cette discipline, à l'épistémologie ainsi qu'aux théories d'apprentissage.

Le premier de ces problèmes est issu de la nature abstraite des mathématiques qui exige le recours à la symbolisation: cette nécessité conduit souvent à une insistance trop vive et prématurée sur la notation mathématique, au point de bien souvent la priver de sens véritable (Ginsburg, 1977; Erlwanger, 1973; 1975). Le contenu mathématique est alors confondu avec quelques signes et l'on méprend la manipulation correcte de ces signes pour la compréhension des transformations mathématiques qu'elle exprime. Ce premier problème est aggravé par le second, l'influence très grande du formalisme: cette influence se traduit dans une recherche souvent exagérée de la précision du vocabulaire et de la rigueur de l'expression de même que dans une approche logico-déductive des concepts mathématiques. Cette vision formaliste présente à l'élève la réflexion mathématique sous la forme statique de résultats achevés sans en révéler les aspects dynamiques pourtant nécessaires à un apprentissage intelligent (Browder et MacLane, 1978; Davis et Hersh, 1980). Le troisième problème touche les conditions mêmes de l'acquisition des connaissances: c'est la présence envahissante des théories behavioristes d'apprentissage où l'accent est davantage porté sur la transmission des connaissances que sur leur construction, l'ensemble de ces connaissances à acquérir - cela est particulièrement évident lorsqu'elles sont décrites sous forme d'objectifs - apparaissant essentiellement comme une liste d'habiletés, de savoir-faire à maîtriser sans qu'elles soient fondées sur les schèmes conceptuels qui les sous-tendent.

Ces trois questions touchant respectivement la nature des mathématiques, les tendances formalistes et les théories behavioristes d'apprentissage ne sont pas étrangères à l'émergence d'un autre problème, d'ordre pédagogique celui-là: la concentration d'un grand nombre d'enseignants sur les réponses de leurs élèves. Ces réponses deviennent souvent les seuls critères sur lesquels s'appuient les maîtres pour juger à la fois de la compréhension et du savoir-faire des enfants. Une bonne réponse peut pourtant émerger d'une démarche erronée et une pédagogie qui s'arrête à la seule production de bonnes réponses poussera bien souvent l'enfant à apprendre par coeur une foule de règles qu'il ne peut justifier ni même s'expliquer: simplement parce que son expérience lui dit que les bonnes réponses, fournies rapidement par ces règles, sont récompensées. Malheureusement, ces règles risquent de demeurer des atomes de connaissance éparpillés et leur nombre peut rapidement dépasser les limites de la mémoire. Et puis, comment l'application mécanique de ces seules règles à des situations bien spécifiques pourrait-elle développer la créativité, engendrer un

esprit d'invention suffisant pour permettre d'affronter des situations nouvelles ou, d'une façon plus globale, amener une véritable évolution des connaissances de l'individu?

QUATRE PROBLEMES LIES A LEURS SOURCES.

Cette centration des maîtres sur les bonnes réponses des étudiants provient d'abord, avons-nous annoncé, de la nature même des mathématiques ou, plus précisément, de la nature traditionnellement reconnue à la mathématique dans le monde scolaire: si l'on excepte la période dite des mathématiques modernes, c'est l'aspect modélisation du réel qui a surtout dominé dans la mesure où l'on y a avant tout considéré les mathématiques comme un outil donnant un pouvoir véritable sur ce réel. Or le propre d'un bon outil, c'est son efficacité dont on juge d'abord à partir des résultats qu'il permet d'obtenir. En ce sens, les règles et algorithmes, s'ils ne constituent pas des modèles de la réalité au sens habituel du terme, n'en demeurent pas moins des outils intéressants car, relativement simples à acquérir ou à faire acquérir, du moins sous forme d'automatismes, ils sont susceptibles de fournir rapidement des résultats pratiques très satisfaisants. Ces résultats deviennent alors le critère ultime permettant de juger à la fois de la valeur de l'instrument et de l'habileté acquise par celui qui le manipule.

Mais la nature des mathématiques n'est pas la seule cause de cette centration des maîtres sur les réponses. De même que le problème de centration n'est pas le seul point de convergence des sources de difficultés énumérées. Car si le problème d'une symbolisation souvent prématurée provient d'une part de la nature abstraite des mathématiques, la philosophie formaliste ou plus précisément néo-formaliste (Dionne, 1982) qui réduit les mathématiques à un jeu de combinaison de symboles manipulés suivant des règles arbitraires, n'y est certes pas étrangère. Comme elle n'est pas étrangère à la confusion entre forme et contenu puisque, ultimement, elle fait de la mathématique une forme sans contenu, l'interprétation des symboles manipulés, la signification que l'on peut leur attribuer, étant repoussées vers ce que l'on appelle la métamathématique. Dans la mesure où, dans leur conception de la discipline mathématique, les maîtres adhèrent ou se laissent inspirer par cette vision philosophique, ils accordent plus d'importance aux aspects syntaxiques qu'aux aspects sémantiques de la mathématique: cela peut se traduire, entre autre, par un enseignement de "règles sans raisons" comme les décrit Skemp (1976), règles dont la seule justification est leur efficacité à produire de bonnes réponses.

Enfin, cette centration des maîtres sur les réponses de leurs élèves est aussi le fait des théories behavioristes de l'apprentissage: surtout si par "réponses", on entend tout autant réaction face à une situation ou un problème donné que la valeur numérique obtenue d'une opération arithmétique par exemple. Une bonne réponse signifie ici à la fois juste reconnaissance de la règle à appliquer, utilisation correcte

de cette règle et exactitude du résultat obtenu. L'enseignement des mathématiques, dans cette perspective behavioriste, s'apparente fort à un conditionnement, la matière apparaît strictement comme un bagage de connaissances à transmettre, une collection de produits finis à accepter et à utiliser comme ils sont. Cette orientation est, notons-le, accentuée par la nature de la discipline qui pousse à travailler très vite aux niveaux abstrait et formel en négligeant trop souvent les étapes préliminaires essentielles à une véritable reconstruction des notions.

Ces quelques remarques nous permettent de constater jusqu'à quel point la philosophie des mathématiques, la discipline mathématique elle-même et les théories d'apprentissage conjuguent leurs influences à la source des problèmes que nous avons brièvement décrits.

COMMENT SE PERPETUENT CES PROBLÈMES.

Ces problèmes sont chroniques avons-nous dit, ayant marqué toutes les époques de l'enseignement des mathématiques, chacun s'y manifestant peut-être avec une acuité particulière et des modalités différentes suivant les périodes mais demeurant sans cesse présent sous une forme ou sous une autre. Quels véhicules leur ont ainsi permis de traverser temps et réformes pour ainsi parvenir jusqu'à nous? Deux véhicules principaux sont, selon toute vraisemblance, à la clef de cette perpétuation: ce sont d'une part les programmes où, dans une large mesure, sont fixés à la fois les contenus de l'enseignement et les approches utilisées et, d'autre part, les enseignants, véritables maîtres d'œuvre de la formation mathématique donnée par l'école.

Le premier véhicule d'importance majeure est donc constitué des multiples programmes d'enseignement des mathématiques qui se sont succédés à travers les époques et les réformes. Ce qui n'est guère surprenant si l'on considère les sources de ces programmes, sources analysées d'un point de vue général par Taba (1962) ou, en s'attachant plus spécifiquement au domaine mathématique, par Callahan et Glennon (1975) et plusieurs autres. Ces sources, qu'elles soient psychopédagogique, sociologique ou mathématique, sont dans une large mesure, celles-là même que nous avons reconnues à l'origine des problèmes que nous avons décrits. Ainsi, la source mathématique joue un rôle non seulement au niveau de la description des contenus mais aussi, facteur plus important encore, au niveau de la conception de la discipline qui est véhiculée: en ce sens, cette source est donc tout autant liée à la philosophie des mathématiques qu'à la discipline mathématique proprement dite.

Le second véhicule majeur de ces problèmes, ce sont les enseignants. Surtout si l'on s'arrête au niveau élémentaire qui nous intéresse plus particulièrement ici, on constate que ceux-ci sont victimes d'une formation trop souvent insuffisante et mal adaptée de sorte qu'ils sont rarement en mesure de résister aux croyances et perceptions de leur époque et de leur milieu. Beltzner et al. (1977) le reconnaissent,

les maîtres du primaire ont, dans nombre de cas, éprouvé des difficultés en mathématiques alors qu'ils étaient sur les bancs de l'école et plusieurs d'entre eux, ajoutent Robitaille et Dirks (1982), n'ont pratiquement reçu aucune formation académique en mathématiques dépassant le niveau de l'école secondaire. Or les programmes de formation des maîtres n'ont pas permis de pallier à ces manques car, rapportent encore Beltzner et al. (1977), on n'est pas arrivé à établir des programmes donnant à la plupart des enseignants une compréhension claire des mathématiques qu'ils auraient à enseigner. Avec comme résultat, constatent-ils, que "dans les écoles élémentaires, les instituteurs ... ont aujourd'hui pour mission d'aider les écoliers à comprendre des choses qu'eux-mêmes n'ont jamais bien saisies". Ces programmes visent, il est vrai, à préparer des généralistes et se contentent, nous dit Gaulin (1978), de fournir, au plan mathématique, un peu de contenu et quelques notions de méthodologie; et, ajoute-t-il, tant au niveau de la formation initiale que dans le perfectionnement, les cours demeurent orientés vers le produit, délaissant les processus caractéristiques de l'activité mathématique. Ces déficiences de la préparation des maîtres, aussi reconnues par Freudenthal (1977), Colmez (1979) et d'autres, permettent d'expliquer leur impuissance à régler ou à dépasser les problèmes que nous avons décrits: ne trouvant que trop rarement dans leur formation les éléments susceptibles d'accroître suffisamment leurs connaissances et de modifier substantiellement leur conception de la mathématique et de son apprentissage, ils demeurent bien souvent le reflet fidèle de leur époque et de ses croyances, tant au niveau de la discipline que de la didactique de cette discipline.

DES ESPOIRS DE SOLUTION?

On l'aura constaté, ces problèmes ne sont certes pas faciles à régler: les facteurs se conjuguant à leurs sources, les éléments contribuant à les perpétuer empêchent d'espérer une réponse simple, une solution-miracle. Un changement réel et profond de l'enseignement des mathématiques demande un effort qui dépasse les modifications superficielles dans les appellations et les façons de faire pour atteindre les personnes concernées et leurs attitudes, toucher les intervenants au niveau même de leur conception des mathématiques et de leur enseignement (Howson, 1979). La question se pose donc: quels types d'interventions auraient le plus de chances d'amener un renouvellement en profondeur de l'enseignement des mathématiques, d'atténuer sinon de faire disparaître ces difficultés auxquelles cet enseignement se heurte constamment malgré la volonté de changement manifestée dans les réformes qui se sont succédées depuis quelques décennies?

Le premier de ces points de contact naturels par lequel introduire une nouvelle conception à la fois des mathématiques et de leur enseignement, c'est le programme ou, plus globalement, le curriculum. Ce biais est d'autant plus évident qu'il émerge pour une bonne part, nous l'avons déjà observé, des éléments que nous avons reconnus à la source des problèmes. Malheureusement, ce biais ne serait en rien

suffisant pour assurer le succès de notre entreprise car si, dans les réformes successives du curriculum, on a tenté d'améliorer l'enseignement des mathématiques de diverses manières, les problèmes de fond n'ont pas été réglés pour autant. D'aucuns pourraient rétorquer que l'on a peut-être pas su faire les bonnes transformations, les réformes adéquates. Mais nous sommes persuadés que la question est ailleurs et que suivant ce que propose Colmez (1979), il faut plus qu'un changement de contenu, plus qu'une réorganisation de celui-ci: il faut arriver à un changement de conception vis-à-vis du savoir mathématique et pour cela toucher les premiers intervenants, les personnes d'abord concernées, celles qui véhiculent ces conceptions auprès des enfants.

LE RÔLE CENTRAL DES ENSEIGNANTS.

C'est ici que le rôle des enseignants prend toute son importance: parmi les personnes concernées, ils seront le pivot de tout renouveau authentique dans la mesure où ils sont les artisans véritables de l'enseignement des mathématiques. Colmez (1979) le reconnaît, "le rôle du maître est fondamental dans l'enseignement élémentaire, plus encore qu'aux autres niveaux de l'enseignement". Et il poursuit en affirmant que le succès d'une réforme dépend largement de ces maîtres, de l'importance que l'on accorde à leur préparation, tant au niveau du contenu de l'enseignement que de ses objectifs et de ses méthodes. Ce que confirme Howson (1979):

"le fait le plus frappant mis en évidence par vingt ans de développement du curriculum est le rôle crucial du professeur. Quelqu'exceptionnels que soient l'équipe ou les documents du projet, son succès dépendra en fin de compte de la réceptivité et de la faculté d'adaptation du professeur".

On retrouve là des évidences dont on a malheureusement guère tenu compte dans les diverses réformes. Kotasek (1972) le rapporte,

"on pourrait affirmer que la formation des maîtres devrait être la fer de lance du développement du système éducatif et que la priorité devrait lui être donnée dans les mesures sociales ou éducatives prises par les Etats. Or l'histoire de l'Éducation fait apparaître que dans la pratique, elle a jusqu'ici été traitée *ex post facto*".

Que la formation des maîtres soit élément central d'un renouveau profond n'a rien de surprenant si l'on considère que peu importe la nature et l'importance des changements proposés au niveau du curriculum, peu importe la compétence des gens qui préparent ces changements, peu importe le sérieux des recherches sur lesquels ils s'appuient, c'est finalement par l'intermédiaire des enseignants que ces changements deviennent réalité, qu'ils sont actualisés dans la pratique scolaire. Et ce n'est que dans la mesure où ces maîtres sont prêts à recevoir ces changements c'est-à-dire qu'ils ont non seulement les compétences mathématiques et psycho-pédagogiques requises mais aussi une perception nette de ces compétences et la certitude que les transformations proposées sont importantes et adéquates qu'ils peuvent modifier en profondeur leur enseignement pour l'accorder aux exigences nouvelles.

LE MANQUE D'INTÉGRATION DANS LA FORMATION DES MAÎTRES DU PRIMAIRE.

La préparation inadéquate des enseignants que nous avons reconnue plus haut est sans doute l'un des facteurs des succès mitigés obtenus lors des réformes des dernières décennies. Colmez (1979) le reconnaît, on a davantage compris "la nécessité de prévoir pour les maîtres une formation relative aux contenus qu'une formation en matière de méthodes". Et encore, cette distinction entre méthodes et contenus ne fait qu'effleurer les questions de fond. Car les méthodes et techniques ne sont que le reflet d'une ou plusieurs théories d'apprentissage qui demeurent habituellement implicites alors que du côté des contenus, le problème vient de ce que, suivant les modèles traditionnels sans cesse reproduits, on se limite à la transmission des notions mathématiques en négligeant de se pencher sur la pensée et l'activité mathématique. On vise ainsi presque exclusivement l'enrichissement des connaissances mathématiques sans réussir ni même chercher à modifier, chez les enseignants, la perception, le plus souvent algorithmique et formelle, qu'ils ont de ces mathématiques. Celles-ci demeurent ainsi une discipline isolée, coupée des autres composantes de la formation des maîtres.

Ceci nous amène d'ailleurs à la critique la plus fondamentale que l'on puisse faire de ces divers programmes de formation: on n'y retrouve guère de véritable intégration de leurs différentes composantes - composantes psychologique, pédagogique, épistémologique et mathématique - ces programmes apparaissant comme une simple juxtaposition d'éléments disparates présentés par un groupe hétérogène de personnes (Gaulin, 1978). Comme le faisait remarquer Freudenthal (1977), on espère que le maître réussira une intégration que ses professeurs ne parviennent pas à réaliser dans leur enseignement.

A cause de ce manque d'intégration, les programmes ne permettent finalement pas de régler le problème épistémologique fondamental qui affecte l'enseignement et l'apprentissage des mathématiques: plutôt que d'être de simples objets à transmettre, les connaissances mathématiques doivent comme les connaissances dans les autres domaines, être construites par l'apprenant. Mais pour arriver à vivre cela dans sa classe, le maître a bien souvent besoin de modifier sa conception des mathématiques qu'il perçoit comme une discipline rigide où règnent l'ordre et la rigueur d'un produit fini, davantage domaine d'habiletés que d'idées (Easley, 1978), pour arriver à une perception moins statique où la dynamique de l'acte de connaissance est davantage présente.

BREVE ESQUISSE D'UNE RECHERCHE EN COURS.

Ce qui précède montre à quel point les quatre grands problèmes décrits trouvent un point d'ancrage important chez les enseignants du primaire et à quel point leur formation n'est pas étrangère au phénomène: que ce soit au niveau de la conception

que ces enseignants véhiculent de la discipline, de la vision philosophique qu'ils en ont, des théories d'apprentissage ou de leur centration sur les réponses de leurs élèves, cette formation ne leur permet pas d'échapper aux problèmes, allant jusqu'à les y plonger par ses insuffisances, dont le manque d'intégration des éléments qui la composent. L'espoir d'une solution à ces problèmes passe forcément par une approche de la formation des maîtres qui modifie la perception qu'ont ces derniers des mathématiques et de leur apprentissage.

A cet égard, une étude exploratoire conduite auprès d'un groupe d'enseignants en perfectionnement s'est avérée extrêmement prometteuse (Gierscovics et Bergeron, 1982a et 1983). Au cours de cette étude, nous avons utilisé une approche centrée sur l'analyse des concepts mathématiques de l'élémentaire à l'aide de modèles de la compréhension: plutôt que d'aborder les notions au seul plan mathématique avec une question comme "qu'est-ce que le nombre?" ou nous arrêter à des questions générales sur la compréhension, nous avons plutôt réuni ces éléments dans des interrogations du type "que veut dire comprendre le nombre?", ces interrogations étant amenées à l'intérieur d'un cadre de référence constitué par ces modèles décrivant la compréhension. Cette façon de faire permet d'intégrer les aspects psychologiques, pédagogiques et mathématiques vers une nouvelle perception de la matière. En effet, la notion étudiée reste présente comme notion mathématique car il serait certes difficile de fouiller la compréhension d'une notion mathématique en faisant abstraction de cet aspect. Cependant, le cadre à l'intérieur duquel s'effectue le travail de réflexion, mettant l'accent sur les aspects cognitifs, amène aussi l'enseignant à s'interroger sur les processus de pensée qui conduisent à la connaissance de la notion abordée. Car décrire, par exemple la compréhension du nombre, n'est-ce pas aussi réfléchir à la manière dont se bâtit cette compréhension? Et le maître peut-il alors éviter de penser à la façon d'organiser les apprentissages et ses interventions dans ces apprentissages en tenant compte de ces réflexions? C'est justement ce que nous entendons par intégration des aspects psychopédagogiques et épistémologiques aux aspects plus spécifiquement mathématiques.

Nous en sommes actuellement à reprendre de façon plus définitive cette expérience exploratoire afin d'étudier plus à fond, à l'aide d'instruments mis au point à cet effet, dans quelle mesure l'approche décrite amène les enseignants à d'une part se décentrer de la réponse de l'élève pour s'attacher davantage à son raisonnement et à évoluer d'autre part vers une perception plus constructiviste des mathématiques et de l'enseignement et de l'apprentissage dans cette discipline.

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TEACHER'S CHARACTERIZATIONS OF STUDENT'S PROBLEM SOLVING EPISODES

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RESUME

Ce texte décrit une étude portant (1) sur les comportements de résolution de problèmes appliqués, utilisés par des maîtres du début du secondaire (middle school) en formation initiale, lors de la solution de quatre problèmes dans le cadre de notre projet sur la résolution de problèmes de mathématiques appliquées, (Applied Mathematical Problem Solving) et (2) sur les relations entre les habiletés en résolution de problème d'un enseignant et son habileté à caractériser avec précision les phases de résolution de problème chez les enfants du début du secondaire (Middle school children). Les problèmes, déjà décrits dans des publications antérieures du projet AMPS étaient: le problème de charpenterie (Lesh et Zawojewskii, 1983), le problème du papier peint (Lesh, 1983), le problème de "Camp Placement" (Lesh 1982) et celui de "Math Placement" (Bell, Fuson et Leah, 1976, p. 530). Chacun des quatre problèmes entraînait l'utilisation de concepts mathématiques élémentaires et chacun constituait la simulation d'une situation qui pourrait facilement se produire dans la vie quotidienne des étudiants ou de leurs familles. Les solutions demandaient, en général, entre 15 et 40 minutes.

Les résultats de notre étude ont montré que les maîtres (particulièrement ceux ayant obtenu relativement peu de succès dans leurs efforts pour résoudre les problèmes) tendaient à imputer aux élèves qu'ils observaient, dans les caractérisations qu'ils en faisaient, leur (non) compréhension et leur(s) (absences) d'habileté propres. Les descriptions fournies par ceux qui réussissaient à résoudre les problèmes se sont avérées plus précises et pénétrantes; ils ont relevé plusieurs types différents de facteurs influençant le comportement des élèves et ont décrit des relations plus complexes entre ces facteurs. Nous croyons que les vidéo-critiques fournissent un contexte à la fois très intéressant et utile pour la recherche sur la résolution de problème et constituent également un puissant outil d'enseignement.

The Rationale:

Our study was motivated by the following observations:

(1) For the kinds of applied problems used in our AMPS project, successful problem solvers tend to think at more than one level; they not only think, they think about thinking. The ability to be (simultaneously) the "do-er" and the "observer" is critical to the solution of many multi-stage applied problems. Consequently, because it is important for students to become aware of their own "meta-cognitive" behaviors (e.g., planning, monitoring, assessing), it is important for teachers to be able to identify these behaviors when they occur naturally in the problem solving activities of students.

(2) Teachers not only teach content, they also implicitly transmit attitudes and understandings about mathematics, and about what it is like to do problem solving. Whether it is desirable or not, students treat teachers as models of "correct problem solving behaviors." Therefore, as teachers "act out" or demonstrate solutions to problems, it is especially important for them to be able to reflect on their own problem solving behaviors, and to help students "see" their meta-cognitive activities.

(3) The ability to accurately and insightfully observe ones own problem solving behavior is closely related to the ability to accurately and insightfully observe and describe/critique the problem solving behaviors of others (or oneself on a video-tape). For example, Vygotsky (1962) predicts that understandings and capabilities that eventually are mastered on an intra-personal

level are first acquired on an inter-personal level. Similarly, Piaget (e.g., 1976) describes ways that inter-personal interactions can lead to intra-personal conceptual development. For Piaget, peer-group interactions may induce cognitive conflict (i.e., the propellant for cognitive adaptation) by confronting students with their own tendencies to "center" (i.e., to notice only the most salient pieces of information, ignoring other important but less salient information) and to be "cognitively egocentric" (i.e., to distort "objective" information and relationships to coincide with pre-conceived "subjective" understandings).

While Piagetians predict that inter-personal interactions can facilitate intra-personal conceptual development, the ability of a student (or a teacher) to recognize undistorted and complexly related details in the behaviors of others will be limited by counter-productive cognitive characteristics associated with primitive conceptualizations of a problem solving situation, or primitive conceptualizations of mathematical ideas underlying the problem solving situation.

(4) Describing and critiquing video-tapes of oneself (or others) provides a powerful tool for both learning and assessment. For example, in teacher training it has long been recognized that: (a) the ability to analyze and assess video-taped "teacher decision-making (i.e., teaching)" sessions is an excellent measure of future teaching performance, and (b) video-critiques also furnish excellent vehicles to help pre-service teachers (or even veteran teachers) improve their teaching and instructional capabilities.

Because of similarities between the mechanisms that are important in problem solving and those that are important in teacher decision-making, we anticipate that video-critiques can play an important role in problem solving instruction. Details about some of these functions have been discussed elsewhere (e.g., Lesh, 1981; Lesh, 1983).

The Study:

Problems: The problems used in this study have been described in past AMPS publications; they were: the Carpentry problem (Lesh & Zawojewskii, 1983), the Wallpaper problem (Lesh, 1983), the Camp Placement problem (Lesh, 1982), and the Math Placement problem (Bell, Fuson, & Lesh, 1976, p. 530). Each of the four problems involved the use of elementary mathematical concepts, and each was a simulation of a situation that might reasonably occur in the everyday lives of students or their families. Solutions typically required 15-40 minutes. Both the students and the teachers worked on the assigned problems in groups of three.

Subjects: Because of time limitations at the Montreal PME meeting, and space limitations in this article, our presentation will focus on detailed transcripts of only three pre-service teachers who were under-graduate students enrolled in a course entitled "teaching Mathematics in Middle School" at Georgia State University. The teachers, Chuck, Caran, and Magda, had six weeks of on-campus instruction in the course followed by four weeks of field experience in a middle-school.

This paper addresses Chuck, Caran, and Magda's performance on the Carpentry problem, and their interpretations of a group of

three students who worked on the Wallpaper problem. For descriptions of other student's problem solving capabilities, and observational capabilities, the reader must be referred to several more lengthy AMPS reports that are currently being written (e.g., Lesh & Landau, note 1).

Background Measures: The following background data were collected on each pre-service teacher in the course. To assess the pre-service teachers' perceptions about problem solving, the Teacher Problem Sort Task (Webb, 1977) was administered; a survey also was given asking what kinds of problems the teachers would prefer to teach, and what kinds of problems they would predict to be most interesting to middle-school students. Measures of content knowledge were derived from scores on course exams covering mathematics, methods, and materials. For teaching performance ratings during the four-week field experience, the instructor used the Classroom Procedures component of the Teacher Performance Assessment Instrument for the State of Georgia (Okey, Capie, & Johnson, 1978). Ratings on interpersonal skills between teachers and their students were obtained using the Interpersonal Skills component of the State materials.

Data Collection: Three major data collecting activities constituted the central focus of the study: (i) video-tapes of the teachers problem solving sessions were transcribed and analyzed, (ii) video-tapes were made of the students' problem solving sessions, and (iii) transcripts were made of the teachers' audio-taped description/critiques of the middle-school students' problem solving sessions.

The students' sessions were viewed by the teachers in a whole-class setting, with each individual teacher recording "notes" using a simplified version of the "Participant/Observer De-Briefing Form" developed by the AMPS project (see Lesh, 1983 for details). The de-briefing form provided categories to encourage observers to notice a variety of different kinds of factors, ranging from specific mathematical skills, or general problem solving heuristics and strategies, to social factors or group dynamics, or metacognitive activities. The teacher de-briefing sessions were held within 24 hours following day that the teachers viewed the students' video-taped session.

On four successive weeks, teachers were assigned to one of the following four activities: (1) Work on the Camp Placement problem and describe/critique students working on the Math Placement problem. (2) Work on the Math Placement problem and describe/critique students working on the Camp Placement problem. (3) Work on the Carpentry problem and describe/critique students working on the Wallpaper problem. (4) Work on the Wallpaper problem and describe/critique students working on the Carpentry problem.

Analyses: Analyses of problem solving sessions, as well as analyses of the description/critiques of problem solving sessions, focused on modeling processes, use of representations, metacognitive activities and understandings, mathematical concepts and understandings, group dynamics, solvers' "individual difference" characteristics, etc., just as in transcript analyses used in other components of the AMPS project.

The Results:

The results of our study showed that teachers (especially those who were relatively unsuccessful in their own problem solving efforts) tended to impose their own (mis)understandings and (dis)abilities into their characterizations of the students they observed. The descriptions given by successful problem solvers were far more accurate and insightful; they noticed more different kinds of factors that influenced students' behaviors, and they described more complex relationships among factors.

The Conclusions:

We believe video-critiques provide both a very interesting and useful context for research on problem solving, and also a potentially powerful instructional setting.

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EDUCATING MATHEMATICS TEACHERS:
THE COGNITIVE PROCESS/CONSTRUCTIVIST PERSPECTIVE

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RÉSUMÉ

En dépit de l'abondance des recherches fondamentales sur les processus cognitifs sous-jacents à l'apprentissage mathématique, les modèles susceptibles de former les enseignants à interpréter pour la classe les résultats de la recherche ont émergé très lentement. L'approche et les activités qui doivent être mises sur pieds dans le cadre du programme de perfectionnement des maîtres (Mount Holyoke College Summermath) conjointement avec le cours d'été 1983 (Summermath) peuvent fournir une aide précieuse pour le développement de tels modèles.

Le programme "Summermath for Teachers" consiste en deux sessions de deux semaines chacune et est offert aux enseignants en mathématiques du secondaire, conjointement avec le programme "Summermath", par l'intégration de la philosophie constructiviste, de la recherche sur les processus cognitifs dans l'apprentissage de la mathématique et des techniques d'entrevue clinique, l'approche veut fournir aux enseignants un cadre de référence susceptible de promouvoir chez les étudiants une meilleure compréhension des concepts mathématiques. Les enseignants seront amenés à considérer l'influence dramatique, qui peut découler d'une vision des étudiants en tant que participants actifs dans la construction des concepts mathématiques, sur la pratique enseignante et sur l'élaboration de curriculum. La démonstration sera faite de l'utilité d'une perspective constructiviste/cognitive pour encourager la formation mathématique de jeunes femmes d'une part, et pour miser sur la pertinence des micro-ordinateurs et du LOGO pour l'amélioration du curriculum de mathématique au niveau secondaire.

L'accessibilité du programme "Summermath" comme milieu - laboratoire, où les enseignants pourront observer la mise en oeuvre effective de la perspective constructiviste-cognitive au niveau secondaire, est une caractéristique nouvelle du programme. Des ateliers incluant des chercheurs et des praticiens permettront aux enseignants d'enrichir encore plus leur compréhension du processus par lequel s'opère la traduction de la recherche à la pratique. L'évaluation de l'enseignant sera faite pendant le programme et un suivi se prolongera dans leurs classes jusqu'au mois de septembre.

La présentation comprendra des descriptions de l'approche et des activités du programme, et traitera de leur efficacité relative pour habiliter les enseignants à introduire dans leurs classes la perspective constructiviste-cognitive.

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Mount Holyoke College's SummerMath for Teachers program was initiated in the summer of 1983, and was offered in conjunction with SummerMath, the College's program for young women in high school who have experienced difficulty with mathematics. The program consisted of two two-week institutes for experienced secondary school teachers of mathematics. The approach integrated constructivist philosophy and research on cognitive processes in mathematics learning in providing teachers with a perspective for promoting better student understanding of mathematics. Teachers considered how a view of students as active participants in the construction of mathematical concepts has strong influence on instructional practice and curriculum design. The usefulness of a constructivist/cognitive process perspective in encouraging the mathematical preparation of young women, and in capitalizing on the appropriateness of microcomputers and LOGO for enhancing the secondary mathematics curriculum, was demonstrated within the subtheme of each session. The SummerMath program was available in a laboratory setting, enabling teachers to observe an effective implementation of the perspective at the secondary level. Course activities were designed to explicate the perspective. Preliminary evaluation indicates that the program was well received by the teachers. They reported the intensive interaction with other colleagues to be particularly valuable. The program seemed to provoke re-evaluation of teaching styles and philosophies.

RATIONALE AND OVERVIEW

Despite an abundance of basic research on the cognitive processes underlying mathematics learning, models for educating mathematics teachers about the research and its implications for classroom practice have been slow to emerge. The work of Herscovics and Bergeron (1980), Bergeron, Herscovics and Dionne (1981), Brown, Brown, Cooney and Smith (1982) and Finkel and Monk (1983) represents a new direction for mathematics teacher education. These researchers share the constructivist view that students arrive at mathematical knowledge through their transactions with instruction and curriculum. Mathematics teacher education programs, focusing on techniques and methods, tend not to address the implications of such a perspective for classroom practice. Given the general concern about the inadequacy of student understanding in mathematics (NAEP, 1983), it is important to offer teachers ways of interpreting the absence of conceptual understanding and the over-reliance on memorized, rote procedures which they encounter in their students.

The goals of SummerMath for Teachers were to convey a theoretical perspective, grounded in constructivism and cognitive processes research, which could provide a coherent interpretation of mathematical learning problems, and which could guide mathematics teaching. To meet these goals there were several objectives: To uncover teachers' initial beliefs and conceptions about mathematics learning and teaching; to demonstrate the disparity between what is taught and what is learned; to provide strategies to help render learning more visible; to interpret the source of mathematical learning problems; to introduce cognitive processes research; and, to encourage careful reflection about mathematics learning and teaching.

A crucial feature of the program was the availability of SummerMath, a six-week program devoted to improving the math understanding of high school women, as a laboratory example that demonstrates the feasibility of applying the constructivist perspective to mathematics education. Twenty-six secondary school mathematics teachers

participated in the program. Most had several years of teaching experience and represented independent, parochial and public schools from across the country.

PROGRAM DESCRIPTION

Articulation of Belief Systems. Beginning with the application process, the focus was directed explicitly towards searching out and exploring the ramifications of one's conceptions about mathematics, teaching and learning for the actual practice of education. An essay required for admission asked them to discuss the attitudes and characteristics they wished to develop in their students. An opening questionnaire inquired into their conceptions of mathematics, teaching, and learning. Initial course activities were designed to illustrate that in teaching and learning mathematics, what is taken for granted often can be problematic. Two videotapes of mathematics teachers, judged by their department chairs as exemplary, were prepared prior to the program. The tapes demonstrated reasonably clear and seemingly adequate examples of direct instruction in algebra and geometry. Immediately following the taped classes, two students from each class were given clinical interviews based on class content. These interviews revealed the students' active attempts to make sense of the mathematics presented, including their misconceptions and partial understandings. The tapes and interviews were shared with the teachers as a springboard for discussing their beliefs about mathematics teaching and learning as well as to heighten their perception that there is a need for further understanding of the mathematics learning process.

Reflective Awareness: Making Learning Visible. Several techniques and activities were utilized to help make learning visible and to engender reflective awareness. These activities included the teachers' own clinical interviews of students from SummerMath, a workshop on paired problem-solving, and focused observation and discussion of the SummerMath program. In addition, a variety of mathematical problems were posed throughout the course to provide a basis

for introspections on the processes entailed in doing mathematics. In the second session, the teachers' learning of LOGO served as a source for reflections on learning. A common journal, wherein the teachers were asked daily to respond to a quotation or question, also incited reflection and discussion.

Cognitive Processes in Mathematics Learning. It is not adequate to present teachers with a plethora of information pointing to the dismal conclusion that mathematics learning is fraught with misconceptions, partial understandings and rote skills. What is needed are ways to make sense of this data on learning, whether the data is in the form of observations, clinical interviews, or research reports. Sense-making devices, or interpretive schemes, were thus provided. Research on errors and misconceptions was presented, with an emphasis on the logical and rational system of thought underlying those errors (Erlwanger, 1975). In analyzing the teachers' clinical interviews, attention was directed to the aspects of learning made visible by the process, including problem-solving strategies, understanding of concepts, private or alternative meanings for concepts and procedures, metacognitive activities, conceptions of math, and self-knowledge. In addition, the work of Krutetskii (1976) on the nature of mathematical abilities was introduced, and guest researchers were invited to discuss their work.

Constructivist Perspective. The constructivist perspective was presented as a powerful way of making sense of learning and mathematical behavior. To that end, essential Piagetian, developmental and philosophical ideas were explained and appropriate readings assigned. The group also inquired into the nature of concepts, conceptual development and understanding. Activities and discussions based on readings included examining the mechanical versus conceptual aspects of specific topics (e.g., signed numbers, ratios, factoring) and addressing the issue of how much skill or practice is needed before conceptualization is achieved. Features of the history and philosophy of mathematics were highlighted in order to reinforce the view that mathematics is a human construction, and to demonstrate that

some of the same issues faced by students in learning mathematics parallel their historical development.

Translation into Practice. A major concern of the program was to deal with the practical necessities of translating the constructivist perspective into classroom practice. Curriculum examples from the SummerMath program that embody the point of view taken, exploration of the conceptual nature of much of what is viewed and taught as mechanical, explicit consideration of the student's point of view, and aspects of the clinical interview technique were presented as fundamental, if partial, answers. As a final project, teachers were required to draw together ideas and schemes garnered from the constructivist perspective, the clinical interview, inquiry into the conceptual nature of mathematics, and the SummerMath curriculum. The goal of this project was to create a portion of curriculum that would take the student's point of view and current understanding into account, and that would implement methods to encourage operations such as flexibility, reversibility, generalization, reflective awareness, and connections to previous knowledge - operations considered crucial for the construction of rich and powerful mathematical concepts.

The Teachers' Experience of the Program. The primary strength of the program lay in the atmosphere of exploration and reflection which was created. Several features of the planning and situation contributed to this success. Particularly important was the intensity and compactness of the two-week institute format, replete with readings, mathematical problems, journal assignments, lectures, discussions, projects and outside speakers. A second major strength was the presentation of a basically unified philosophy towards mathematics and learning that was exemplified and reinforced at every turn. The curriculum, taken as a concerted whole, engendered an extraordinary degree of self-examination, reflectiveness and questioning in the teachers. In the atmosphere created, it became legitimate to question teaching practices and to rethink the nature of mathematics and learning processes.

Providing a setting for the teachers to compare teaching methods, examine student behaviors, and to question the taken-for-granted aspects of teaching at a safe distance from the politics of the school setting, had a striking sociological impact. Several of the teachers remarked on how pleased they were to get beyond the usual political and practical talk that characterizes teachers' rooms. The fact that most of the teachers had at least five years of teaching experience was also an important factor in terms of group dynamics; the common sentiment was that despite the geographical diversity represented, there was a common ground of experience to unite them.

There was a range of reactions to the two institutes and to their various components. This portion of the paper is based on observations of the teachers throughout the program with respect to their class participation, journal entries, final projects, program evaluations, and mealtime conversations.

In response to the initial videotapes, teachers' comments expressed a spectrum from self-complacency to painful self-examination: "I am now reassured that what I am doing in my classroom is good." "I can recognize myself on those tapes - I wonder if I have such a stupefying effect on my students." The clinical interviews with students from those classes revealed a striking disparity between how material is presented and how students understand that material. This cognitive dissonance contributed to the teachers' willingness to consider seriously the view of cognitive processes explicated in the program. The teachers' reactions to their own clinical interviews brought forth compelling statements of self-analysis. The predominant sense was that it was extremely difficult not to "teach." They were struck by how hard it was to listen, not interrupt, not focus on getting the right answer, and not lead. These reflections transferred to general concerns with their own teaching styles: "How often do I interrupt my students and give them the answers?"

The most potent supplement to the readings and talks was the teachers' own learning experiences and their subsequent reflections and

analyses of their learning. One teacher wrote in her journal that before LOGO she assumed that the learning process was a straightforward one, but that after several bouts with the LOGO turtle she felt learning to be not so well described by the "straight-path" metaphor. She also asserted that she now had far more sympathy for the learning troubles of her students.

CONCLUSION

A two-week period is somewhat restrictive in terms of allowing the time and space for assimilation and accommodation. However, journal entries and final evaluation comments suggest that the combination of course experiences did indeed promote substantive questioning, re-assessment and optimism concerning possibilities for changing the structure of mathematics classes. The teachers conveyed an excitement over their observations of the SummerMath students' persistence in tackling math problems and confronting their understandings in the clinical interview. Ideas for implementing the constructivist perspective in test construction, in approaching concepts in new ways, and in instructional style were spontaneously generated by class members.

SummerMath for Teachers seems to have been a worthwhile attempt at developing a mathematics teacher education program for experienced teachers. Follow-up to be conducted during the 1983-1984 academic year will be essential in undertaking a more complete evaluation of the program.

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LA MINI-ENTREVUE: UN NOUVEL OUTIL D'EVALUATION
DE LA COMPREHENSION EN MATHEMATIQUE AU PRIMAIRE

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ABSTRACT

In the teaching of arithmetic in the elementary school there is often a tendency for the teachers to focus on their pupil's abilities rather than on their thinking processes. But if understanding is to be an objective of the mathematics curriculum, he has to know the evolution of the child's cognition. Written tests are adequate when the aim is to evaluate arithmetical abilities, but can they inform us as to the child's thinking and reasoning? Such information can only be obtained through individual questioning. To satisfy the teachers' needs, as well as the constraints of the classroom, a new version of the clinical interview, that we coined the MINI-INTERVIEW, was developed. This new tool designed to be integrated into the regular classroom activity.

The MINI-INTERVIEW, which lasts between 4 to 8 minutes, aims at identifying the child's thinking processes through questioning related to precise tasks and specific aspects of the understanding of conceptual schemas. An example of such a mini-interview involving counting with small numbers in the context of addition illustrates this proposed form of questioning. Finally evidence is brought forth showing that elementary school teachers can use the mini-interview with all of their pupils while integrating it to their regular arithmetical activities. Moreover, they succeed in constructing mini-interviews on their own related to various arithmetical notions.

(English hand-outs will be available)

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INTRODUCTION

Dans l'enseignement de l'arithmétique au primaire, il y a souvent une tendance de la part des enseignants à se centrer sur les habiletés de leurs élèves plutôt que sur leurs processus de pensée. Il ne s'agit pas nécessairement d'une intention consciente de la part de l'enseignant mais plutôt d'une certaine ignorance des processus utilisés dans la construction d'un schème conceptuel et d'un manque de moyens à sa disposition pour cerner la pensée et le raisonnement de l'enfant. L'élève passe donc une grande partie du temps alloué à l'enseignement de l'arithmétique à répéter des exercices sur les opérations arithmétiques, exercices qui sont, par souci d'efficacité, le plus souvent écrits. De plus, l'élève ne reçoit habituellement comme seul feedback de son rendement que sa copie où la bonne réponse est cochée et la mauvaise marquée d'une croix. Rien d'étonnant alors que les enfants perçoivent l'arithmétique comme simplement des habiletés à acquérir (Erlwanger, 1973, 1975) et non pas des schèmes conceptuels à bâtir, des formes de raisonnement à développer, des relations à établir.

Dans le contexte de la formation et du perfectionnement des maîtres en didactique de la mathématique, on ne peut se limiter à exiger de ces derniers qu'ils se préoccupent de la compréhension en tant qu'objectif général. Car, tant que cet objectif reste général, il demeure à l'état de vœu pieux sans qu'il puisse être intégré à la réalité scolaire de tous les jours. Ce n'est que lorsque l'enseignant parvient à décrire, pour chacun des concepts qu'il enseigne, ce qu'il entend par "comprendre", qu'on peut espérer faire de la compréhension un objectif de l'enseignement. Mais même alors qu'il aura précisé ce qu'il entend par compréhension, il se heurtera aux problèmes de comment la cerner, comment l'identifier, comment l'évaluer.

La psychologie cognitive et la psychologie du développement ont eu un apport considérable tant au niveau de nos connaissances sur la construction des schèmes conceptuels, tels le nombre, l'addition, etc., qu'au niveau des moyens à utiliser pour cerner la pensée et les processus de raisonnement de l'enfant. Ainsi, les méthodes cliniques développées par l'école piagétienne sont probablement les mieux appropriées pour atteindre ce but. Ces méthodes ont été élaborées comme outils de recherche en psychologie et en épistémologie. Cependant, comme les besoins de l'enseignant diffèrent de ceux du chercheur, ces outils d'investigation doivent être adaptés aux besoins du maître dans le contexte scolaire. Mais quels sont ces besoins et quelles sont les contraintes imposées par ce contexte scolaire?

BESOINS ET CONTRAINTES

Dans une perspective constructiviste de l'apprentissage, le maître aide l'enfant à construire ses connaissances (par opposition à une simple transmission de connaissances) lesquelles se traduisent par des schèmes conceptuels dont l'acquisition peut s'étaler sur des périodes fort longues. Bien entendu, avant d'introduire un nouveau concept ou de poursuivre une construction déjà amorcée, l'enseignant a besoin de déterminer la base cognitive de l'enfant à partir de laquelle il peut le faire progresser. Il s'agira alors de déterminer si les notions pré-requises sont présentes, et s'il y a lieu, combler les lacunes et corriger les fausses interprétations. Ce sont justement là trois causes principales de difficultés d'apprentissage pour le plupart des enfants.

S'il est clair que l'enseignant a besoin de connaître l'évolution cognitive de ses élèves, les moyens lui permettant de la suivre ne sont pas évidents. En effet, les moyens couramment utilisés se situent au niveau de tests écrits. Ces derniers sont tout à fait indiqués lorsqu'il s'agit d'évaluer les habiletés arithmétiques mais peuvent-ils nous renseigner sur la pensée et le raisonnement de l'enfant? Quand celui-ci nous répond que "cinq plus quatre font neuf", pouvons-nous savoir si le problème posé a pour lui une signification et que sa

réponse n'est pas le fruit d'une pure mémorisation? Lorsque la question est posée par écrit, non seulement fait-elle intervenir des difficultés additionnelles dues à la représentation symbolique du problème mais de plus, la forme écrite nous empêche de voir si l'enfant compte et surtout comment il compte. De telles informations ne peuvent être obtenues que par un questionnement individuel et ceci suggère comme moyen à utiliser une forme de l'entrevue clinique.

Suggérer un questionnement individuel de chacun des élèves est peine perdue si le maître le perçoit comme une charge additionnelle nécessitant des heures supplémentaires de travail. Il faut donc que cette forme d'évaluation formative soit intégrée à son travail régulier, c'est-à-dire à l'intérieur de la classe. Et ceci impose des restrictions importantes à la forme d'entrevue réalisable. Ainsi, alors que le chercheur dispose de tout le temps nécessaire pour questionner à sa guise et n'est limité que par l'attention de l'enfant, tel n'est pas le cas pour l'enseignant voulant s'occuper d'un seul élève sans ignorer les autres. Ceci exige une organisation appropriée de la classe, organisation d'autant plus facile pour les enseignants ayant déjà une certaine expérience en enseignement individualisé. L'assignation d'un travail individuel au restant de la classe permettra à l'enseignant de questionner un même élève pendant une courte période sans être dérangé; de 4 à 8 minutes, selon l'âge des enfants.

Ce facteur de temps disponible limite le champ d'évaluation. En effet, le chercheur peut se permettre de préparer une entrevue afin d'examiner globalement un schème conceptuel donné tel que le nombre ou l'addition. Par contre, l'enseignant ne disposant que de 4 à 8 minutes se doit de restreindre son évaluation à un aspect précis du schème conceptuel (e.g. la façon d'additionner). L'entrevue du chercheur doit demeurer flexible car il se doit de poursuivre des pistes intéressantes l'amenant à découvrir les processus cognitifs des enfants. Mais la découverte de ces processus est la tâche du chercheur et non pas celle de l'enseignant. L'entrevue dont ce dernier a besoin vise l'évaluation de la compréhension en fonction de ses interventions pédagogiques ainsi que le dépistage des enfants en difficulté. Il

peut donc se permettre une forme d'entrevue beaucoup plus standardisée se limitant essentiellement aux tâches et aux questions appropriées préparées d'avance.

LA MINI-ENTREVUE

C'est pour répondre aux besoins des enseignants et aux contraintes de la classe que nous avons élaboré une forme de l'entrevue clinique que nous qualifions de MINI-ENTREVUE. Tout comme l'entrevue clinique, la MINI-ENTREVUE vise à cerner la pensée et le raisonnement de l'enfant à travers un questionnement se rapportant à des tâches précises. Mais elle s'en distingue par rapport au temps alloué (de 4 à 8 minutes) ainsi que par l'ampleur de l'évaluation, c'est-à-dire qu'elle ne vise qu'un aspect précis de la compréhension d'un schème conceptuel. Bien sûr, ceci suppose que l'enseignant possède au préalable certains critères lui permettant de juger de la compréhension de la notion visée.

La mini-entrevue étant un outil d'évaluation formative, elle doit renseigner le maître sur ce que l'enfant a compris par rapport à ce qu'il a enseigné. Un tel contexte rend possible la préparation d'une courte entrevue dont les tâches et les questions sont élaborées et agencées de façon systématique et rationnelle.

Questionner des enfants n'a rien de nouveau. Tout bon enseignant le fait dans le cours normal de son enseignement. Mais, bien souvent, ceci se limite à des élèves qui manifestent certaines difficultés d'apprentissage, alors qu'on se préoccupe moins de ceux qui fournissent toujours la bonne réponse. Comme la mini-entrevue vise les processus utilisés, elle s'applique à chacun des élèves de la classe, que sa réponse soit bonne ou mauvaise. Ceci n'implique pas que l'enseignant devra passer tout son temps à rencontrer individuellement ses élèves. Car, en effet, il ne s'agit d'utiliser cette forme d'évaluation que pour les notions-clés du programme dont le nombre est assez restreint à chaque niveau du primaire. Ainsi, après une séquence d'enseignement sur l'addition, le maître peut évaluer la compréhension de ses élèves en échelonnant ses entrevues sur deux ou trois semaines

à raison d'une ou deux entrevues par jour.

UN EXEMPLE DE MINI-ENTREVUE

Prenons comme exemple le dénombrement à partir d'un des termes dans une addition comme "six plus trois" où l'enfant compte "six, ... sept, huit, neuf". Fuson et al (1980) ont montré qu'une telle procédure fait appel à un niveau de compréhension assez sophistiqué. En effet, quatre significations complémentaires, toutes nécessaires pour compter à partir du premier nombre, peuvent être rattachées à la répétition du mot "six". D'abord le mot "six" désigne la cardinalité du premier ensemble (le dernier mot prononcé dans la suite des nombres correspond au nombre d'objets); il implique l'anticipation du résultat qui serait obtenu en comptant ("si je dis "six", je sais qu'en comptant les objets à partir de un j'arriverai à six"); de plus, il résume l'acte de compter ("six" est ici une abréviation de "un, deux, trois, quatre, cinq, six"); enfin, il relie les deux ensembles en question en servant de point de départ pour l'énumération du second ensemble. Le dénombrement à partir d'un des termes de l'addition apparaît donc comme une étape importante dans l'évolution arithmétique de l'enfant et vers la fin de la première année ou au début de la deuxième un grand nombre d'élèves y parviennent. Cette mini-entrevue permet de s'en assurer. L'importance d'une telle information sur l'évolution des enfants devient évidente lorsque ceux-ci sont confrontés à des problèmes tels que $16 + 3$ et qu'ils doivent recommencer à compter à partir de un. Voici donc un exemple du questionnement utilisé pour la mini-entrevue touchant le dénombrement dans l'addition des petits nombres.

QUESTIONNEMENT

Question 1: "Peux-tu compter à partir de 6? ... à partir de 12?"

Commentaire: Le but de cette question est de s'assurer que l'enfant peut compter à partir d'un nombre donné. C'est là un pré-requis essentiel pour l'application de la procédure "compter à partir de l'un des termes" dans une situation additive. Donc, si l'élève ne peut pas

compter à partir d'un nombre donné, l'entrevue s'arrête ici.

Question 2: "Peux-tu me dire combien font $6 + 5?$... $12 + 4?$ "

Commentaire: Cette question orale a pour but de déterminer les limites de la mémorisation car ce n'est qu'en la dépassant qu'on peut identifier les procédures utilisées par l'enfant. (Sans recours à l'écriture ou objets).

Question 3.1: "Peux-tu me dire comment tu ferais pour trouver $14 + 3?$ "

S'il ne peut décrire sa procédure, lui poser la question

Question 3.2: "Comment pourrais-tu expliquer à un ami ce qu'il doit faire pour trouver la réponse à $9 + 6?$ "

Décrire sa procédure: a) compte à partir de un b) compte à partir de l'un des nombres c) compte en décomposant et regroupant d) autre

Commentaire: L'addition choisie dans cette question ne doit pas avoir été mémorisée. Si l'élève ne donne aucun signe de la procédure qu'il emploie peut-être lui sera-t-il plus facile de verbaliser sa démarche lorsqu'il a à l'expliquer à un autre enfant. Le choix des nombres est important. En effet, en choisissant le premier nombre assez grand et le 2e assez petit, "compter à partir du premier nombre" pourrait paraître à l'élève une procédure plus efficace que "compter à partir de un". Cependant, même s'il emploie cette dernière procédure, cela ne veut pas dire qu'il ne peut pas employer la première, ce qui sera vérifié par les questions qui suivent. Evidemment, si l'élève décompose et regroupe, il indique une procédure encore plus avancée. Comme les stratégies utilisées peuvent varier selon la représentation du problème (la présence d'objets peut encourager le comptage à partir de un), il importe dans un premier temps de ne pas centrer l'enfant sur une procédure plus primitive mais plutôt de faire appel à la représentation mentale.

Question 4: "Ce matin, je t'ai donné quinze jetons. Combien en as-tu? Ensuite l'après-midi, je t'en donne quatre autres; les voici.

Combien en as-tu en tout?

Décrire sa procédure: a) compte à partir de un b) compte à partir de l'un des nombres c) compte en décomposant et regroupant d) autre

Commentaire: Encore une fois, il s'agit de s'assurer que l'on dépasse

la mémorisation de l'élève et que les nombres choisis incitent au comptage à partir du premier nombre. La présence de jetons devrait rendre le problème plus facile, ce qui permettra d'identifier les stratégies employées par des enfants qui ne pouvaient pas répondre à la question précédente et aussi, pour ceux qui avaient réussi, de comparer les procédures utilisées avec ou sans objets. Le problème choisi illustre la situation d'ajout parce qu'elle semble plus propice à faire intervenir le comptage à partir d'un terme. La situation de réunion viendra plus loin.

(Si l'élève compte toujours à partir de un)

Question 5.1: "As-tu besoin de compter à partir de un?"

(Si non): "Peux-tu me montrer comment tu le ferais autrement?"

(Si oui): "Combien en as-tu toi?" (l'élève les compte au besoin)

"As-tu besoin de recompter les tiens pour savoir combien il y en a en tout?"

(Si non): "Montre-moi comment tu fais pour savoir combien il y en a en tout?"

(Si oui): "Pourquoi?... (10 secondes)... Peux-tu compter combien il y en a en tout en commençant à 15?"

Question 5.2: "Voici un ensemble de seize jetons et un autre de trois jetons. Peux-tu me dire combien il y a en tout?"

(Si l'élève compte encore à partir de un)

Question 5.3: "Qu'est-ce que tu as fait dans le problème précédent pour trouver combien il y en avait en tout?"

Commentaire: Comme on peut s'en apercevoir, ce questionnement doit être subtil. En effet, il s'agit d'être le moins directif possible afin de permettre à l'enfant de réfléchir au problème. Et de fait, ce n'est qu'à la toute fin que nous suggérons la procédure visée. Même si la mini-entrevue vise essentiellement à situer la compréhension de l'élève, l'enseignant peut difficilement rater une aussi bonne occasion lui permettant de guider l'élève à relier le comptage à partir d'un nombre donné à la résolution de problèmes d'addition. La question 5.2 est importante vu qu'elle nous permet de voir si l'élève comptera spontanément à partir du premier terme. Dans le cas contraire, la question 5.3 pourrait l'amener à interioriser la procédure qu'il

a employée auparavant.

Question 6: "Voici un ensemble de douze jetons et un autre de cinq jetons. Maintenant je cache le paquet de douze jetons. Peux-tu me dire combien il y en a en tout?"

Commentaire: Ce problème est posé à tous les enfants. Pour ceux qui comptaient à partir de un (donc qui ont eu à passer par la question 5) c'est vraiment un test nous permettant de voir si nous avons réussi car en cachant le premier ensemble l'élève ne peut trouver la somme qu'en comptant à partir du premier nombre. Même pour ceux qui le faisaient déjà, on peut s'attendre à certaines surprises. Par exemple, certains auront plus de difficulté simplement parce qu'un des ensembles est caché. D'autres, qui décomposaient et regroupaient, pourraient fort bien abandonner cette procédure pour compter à partir du premier nombre.

EN GUISE DE CONCLUSION

Suite à l'élaboration de ce nouvel outil d'évaluation, nous avons voulu savoir si l'enseignant peut l'utiliser dans les conditions normales de la classe. Cette question de faisabilité a été vérifiée auprès de deux groupes d'enseignants du primaire (inscrits dans un programme de perfectionnement en sciences et mathématiques) dont plusieurs ont pu utiliser la mini-entrevue décrite ci-dessus, avec chacun de leurs élèves, en l'intégrant à leur enseignement régulier. La réaction des enseignants et de leurs élèves a été des plus positive. En contrastant les réponses obtenues dans un test écrit avec l'information tirée de la mini-entrevue, les enseignants furent fort surpris de constater que ce nouvel outil leur permettait d'identifier les processus utilisés par les enfants, et qu'ils ne pouvaient plus se fier aux seules réponses données. Bien plus, selon leur propre témoignage, la mini-entrevue a été l'occasion d'apporter à chaque enfant une attention individuelle plus grande avec comme conséquence que plusieurs enfants se sont sentis revalorisés par l'attention qu'ils recevaient. Une deuxième question que nous nous sommes posée visait à savoir si l'enseignant pouvait, par lui-même, préparer une mini-entrevue. Dans le cadre de ce cours, les

professeurs, regroupés en équipes, ont pu élaborer diverses mini-entrevues portant sur divers concepts arithmétiques enseignés au primaire. Cependant, ce travail leur a demandé beaucoup d'effort et a fait ressortir la grande confusion qui existe chez eux entre habiletés et compréhension.

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THE SCHOOL AS A LEARNING ENVIRONMENT FOR TEACHEPS

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RÉSUMÉ

Grâce à l'application de mon nouveau modèle d'intelligence, maintenant je crois bien savoir enseigner les mathématiques, pour la compréhension des relations, aux enfants âgés de cinq à onze ans. Il reste pourtant un problème à résoudre, à savoir, comment amener la majorité des enseignants qui n'assistent pas à des congrès, qui ne lisent ni manuels pour professeurs ni revues concernant l'enseignement des mathématiques, et qui ne suivent jamais de cours de recyclage, à utiliser cette connaissance.

La solution que je vous propose est la suivante; il faut que l'école soit un milieu d'étude tant pour les professeurs que pour leurs élèves. Je vais vous expliquer comment faire pour y parvenir: la méthode en est l'application du nouveau modèle d'intelligence. Je vais vous relater comment cette méthode a été mise à l'épreuve ces deux dernières années.

Some time ago, I had come to realise that the school teachers I was meeting and working with were not a typical sample. They were the ones who came to conferences and seminars, or attended in-service courses: the keen ones. This is a natural result of the traditional framework, in which pupils come to schools to be taught by school teachers (T1s), and T1s come to universities and similar institutions to be taught by T2s. (By a T2 I mean someone who teaches teachers.) This has been the case for the major part of pre-service education, and for in-service education.

More recently, I have come to question whether this is the only pattern, or the best. We learn to swim in the water, not on dry land. We learn to drive in a car. This applies to beginners and advanced students alike. Not only does common-sense suggest that this principle may be a general one, but theory also. If learning is adaptation to an environment, then the greater part of it needs to take place in the environment where its results will be put to use.

Currently, I am developing an approach to in-service education which is based on this hypothesis, that perhaps the best learning environment for school-teachers is a school. For five years now, I have been working on methods and materials for teaching mathematics to children aged 5 - 11, based on my new model of intelligence; and for the last two years, I have

been taking these into schools, both for field testing the activities themselves and also for helping teachers learn to use them. This is how we work together.

Once a week my research associate and I spend a whole morning in a school. During lesson times, we work alongside the teachers in their own classrooms, sitting round a table with a group of about six children. This is a typical teaching situation in many English primary schools. There is, however, a difference, in that the head teacher comes along too. Part of the time she watches what we are doing. This enables her to build up a picture of what is going on throughout the school. Part of the time she relieves the class teacher, so that the class teacher can come and watch. Then at lunch time we and all the teachers come together in the staff room, and talk about the morning's activities over our sandwich lunch. The activities are demonstrated - usually we do them together. All the teachers also have all the written materials which will eventually be published. And we discuss.

It's as simple as that. Nevertheless, what we do together is based on the same model for intelligent learning as the activities with children. This is a constructivist model, which suggests that for constructing our schemas we need to use as many as possible of the following ways. (Skepp, 1979 & 1981.)

SCHEMA CONSTRUCTION

BUILDING	mode	TESTING
Experience	1	Prediction
Communication	2	Discussion
Creativity	3	Internal consistency

I attach great importance to mode 1 building, in the form of first-hand observation by teachers of their own children. If things have gone as predicted, which is to say according to my learning plan for the children - fine. This is mode 1 testing. If not, we have more to discuss (Mode 2), and often learn more when things have not gone altogether smoothly than when they have. There isn't the attitude, "I know, you don't. I'm here to tell you, you are here to learn", which (dare I say it) characterises the attitude of some teachers towards their pupils. We all learn together. Important also is this attitude to mistakes. We must get teachers away from giving children the feeling that mistakes are culpable, being signs of inattention or lack of effort; otherwise children won't seek help when they need it. In this we must ourselves set an example. (Papert's treatment of de-bugging is another good example of this different attitude to errors: see Papert, 1980.)

Some of my own communications are directed towards relating the learning activities to the model, and bringing out points of importance in the

mathematical content. The teachers communicate too, about what has happened since our last meeting during their own trials of the materials, and about any practical problems which have arisen. All this makes good use of Mode 2. These discussions go well, because they are based on observations shared by several of us, and are about their own children as much as about mathematics and psychology.

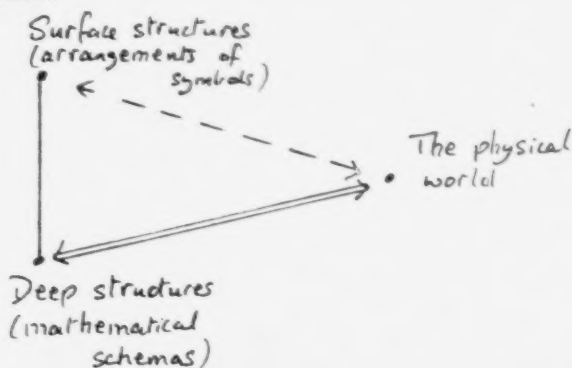
Teachers' own creativity also comes into play (mode 3 building). Often they suggest additions or extensions to the activities; and also extensions made by their children. We discuss these, and naturally want to check that they are consistent with the overall approach, and with the other mathematical content (mode 3 testing). Sometimes these extensions are incorporated into the printed material; but on the whole I try to avoid this temptation, so that space is left for other teachers to personalise the activities for themselves and their pupils.

Thus, the same model for intelligent learning which is used for the children is also used for ourselves, which for me is as it should be.

The materials and activities of the Primary Mathematics Project play as important a part in learning by the teachers as by their pupils. Each of these activities embodies the model, in one or more of its aspects; and collectively they form a demonstration of the model in action. In this way, the teachers' learning is not confined to our

contact time. All the time they are using these activities with their own children, the teachers themselves are learning too. This has been a tremendous bonus, only partially foreseen. For those in whom this process becomes established, professional satisfaction is increased, personal growth starts up again, and there is often spin-off into other areas of teaching also.

This learning takes place in a way not possible with a mainly text-based, pencil and paper approach. The reason is both simple and interesting, and arises from another part of the model.



What makes mathematics so powerful a tool for understanding, predicting, and sometimes controlling events in the physical world is the fact that it affords such an accurate, multi-purpose, and almost ready-for-use model of the physical world. BUT this correspondence is the one shown in the diagram by the lower arrow, not the upper ones: between the deep structures of

mathematics and the physical world, not the surface structures and the physical world.

So when we give children materials to manipulate which embody these deep structures, we are doing two very important things. We are letting them experience the power of mathematics to organise and understand the physical world in their own here-and-now. And when children are working with these manipulatives, they reveal their thinking more clearly than they ever could by words or writing.

With all these advantages, there is one great disadvantage. It is very costly in time of the visiting consultant, the T2. In the present case this is offset by the fact that the T2 himself is learning a lot. It is also, as already explained, very cost-effective in the time of the T1s, and these are more in number.

My own experience has been that in the right kind of schools, this approach succeeds better than any other form of in-service education in which I have been involved. How possible it may be on a larger scale, I have not yet explored; nor in what directions more practical to modify it for wider use. It is particularly in this last area that I shall welcome suggestions from your own experience.

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L. UNDERSTANDING, CONSTRUCTION OF MEANING,
COGNITIVE LEVELS, DIAGNOSIS /

COMPREHENSION, CONSTRUCTION D'UNE SIGNIFICATION
NIVEAUX COGNITIFS, DIAGNOSTIC

LEARNING OUTCOMES IN MATHEMATICS AS INFLUENCED BY INSTRUCTION

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RESUME

Anderson (1980) décrit deux types de résultats résultant de l'apprentissage: la connaissance de faits (déclarative knowledge) et la connaissance procédurale. Les résultats de l'apprentissage qui concernent le "savoir comment" sont des exemples de connaissance procédurale, alors que ceux qui concernent le "savoir que" se rapportent aux connaissances de faits. Est-ce que ces deux types de résultats existent dans l'apprentissage des mathématiques? Si c'est le cas, sont-ils influencés par la nature de l'instruction?

Dans une étude sur l'apprentissage des mathématiques, une analyse factorielle indique que la performance des élèves était caractérisée par deux dimensions dans les résultats de l'apprentissage: l'une identifiée à la performance à calculer et la seconde, à la compréhension des énoncés verbaux. L'instruction qui met l'accent sur le sens des énoncés et les relations entre les connaissances verbales facilite l'acquisition de la connaissance de faits. L'instruction qui se centre sur le contenu, dans l'emploi d'algorithmes et de routines de calcul, facilite l'acquisition de la connaissance procédurale. L'instruction qui met l'accent sur le sens plutôt que sur le développement facilite davantage la rétention, à la fois pour la connaissance de faits et procédures des habiletés. Les résultats de l'étude sont discutés en termes de la compréhension relationnelle et instrumentale de Skemp (1976) et des structures de mémoire et résultats d'apprentissage de Gagné et White (1978).

When teaching mathematics, do we get what we teach for? Do students focus their attention on certain aspects of learning the content such as rote memorization of verbal statements, or computational procedures, or understanding the concepts? Does what they learn depend upon what they perceive the emphasis is in instruction? It is expected that what each student learns from the content is greatly determined by the kind of instruction the student receives. The research hypothesis in this study is that for any topic in mathematics there are several possible kinds of learning outcomes and the particular kind of outcome a student acquires is significantly influenced by the nature of the instruction.

A model of the learning process is hypothesized as shown in Figure 1.

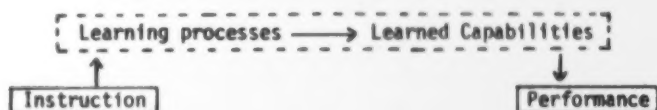


Figure 1. The hypothesized learning model.

Types of learning outcomes which are a result of learned capabilities in particular memory structures are observed in differential performance on specified tasks. Different instructional strategies are presumed to engage different internal learning processes which in turn determine the kind of capabilities that are learned.

For any segment of mathematics we can observe students who perform quite differently with respect to the topic. For example, if the mathematical topic is the formula for finding the area of a rectangle, $A = L \times B$, then we might expect a wide range of learned performances from students. They may (a) commit the formula to memory, (b) justify the formula by showing how it accumulates the units of area, (c) calculate a value for one variable given values of the other two, and (d) apply the formula to finding the area of

figures that in some way can be reconstructed as a rectangle. Although these are only a few of the many performances which may be observed as learned outcomes, these fall generally into two main categories: learning outcomes of "knowing how" and "knowing that." Using the formula as in (c) and (d) belongs in the "knowing how" category. Performances (a) and (b), requiring verbalization rather than computation, fall into the category of "knowing that." The assignment of learning outcomes to these two categories is determined by the types of performance observed. If the observable behavior is predominately in a verbal form such as oral or written justification or explanation, the learning outcome is one of "knowing that" and is classified as declarative knowledge. However, if the resulting performance is computational in nature, the learning outcome is put into the "knowing how" category, called procedural knowledge.

Anderson (1980) notes: "This distinction between knowing that and knowing how is fundamental to modern cognitive psychology" (p. 223). Declarative knowledge is "static" knowledge underlying the recall of facts and tends to be information that can be verbally communicated. It frequently takes the form of mental imagery or abstract propositions. Procedural knowledge is cognitive skills or the abilities to perform various intellectual procedures and may be referred to as knowledge based on algorithms.

According to Gagné and White (1978), these two types of learning outcomes are the results of specific memory structures in which the lasting results of learning are stored. Their model includes four different kinds of memory structures: propositions, images, episodes, and intellectual skills. Although images and episodes do have direct links to specific types of outcomes, they are probably most often supporting memory structures for propositions and intellectual skills. The type of learning outcomes associated with the propositions memory is knowledge stating, synonymous with declarative knowledge. Intellectual skills memory is those capabilities providing for rule application or procedural knowledge

outcomes. Images memory leads to drawing outcomes, allowing for visual representations of content. Episodes in memory would provide the learner the capabilities of identifying action sequences. The four memory structures have direct links with each other and any combination of them is possible.

The information-processing model of learning which incorporates memory structures also includes learning processes which produce these lasting results stored in memory. These processes are cognitive processes which may be either a sequence of mental operations acting on information already in a memory structure or a sequence of internal changes in information (Mayer, 1977). These internal restructuring processes are guided by external factors, of which instruction in the classroom is considered to be an important one. Thus the nature of instruction a student encounters is believed to be a major influence on the type of mathematics learning achieved.

METHOD

Variables

To test the research hypothesis two instructional strategies with different emphases were designed for teaching modular arithmetic. The Verbal strategy emphasized the inter-relationships of the content, providing students opportunities to express verbally their understanding of the relationships. Clocks were used as visual representations of the verbal statements as an additional aid in developing meanings of and relationships between verbal statements. A strategy with this emphasis was hypothesized to produce more learned capabilities in the propositions memory than in other memory structures and thus would lead to greater achievement of declarative knowledge learning outcomes. The clock representations seemed to relate to the images memory structure with drawings as the learning outcomes.

The Computational strategy emphasized the acquisition of procedures and computational proficiency. There was no attempt made to provide

meaning to the routine algorithmic procedures shown in which the procedure to be followed and the correct answer were stressed. This type instruction was hypothesized to produce more learned capabilities in the intellectual skills memory which would result in learning outcomes of procedural knowledge. No other memory structure was directly involved in this strategy.

The dependent variables in the research design were sets of declarative and procedural knowledge learning outcomes. Each of these knowledge dimensions was perceived as a continuum of capabilities ranging from specific to general. For the purpose of the study sets of capabilities representing levels of transfer were defined along each continuum. Since these capabilities as lasting results in memory can only be inferred by differential performance on specified tasks, an initial achievement test and a retention test were constructed with subtests corresponding to the sets of capabilities defined. Figure 2 gives the test plan used.

Level of Transfer				
	I	II	III	IV
Declarative Knowledge	Recall verbal statements as given.	Justify procedures; Apply verbal statements to seen and unseen examples in mods used during instruction.	Relate verbal statements to clock, using mods not seen previously.	Generalize verbal statements; Apply verbal statements in literal examples.
Procedural Knowledge	Solve examples identical to those seen.	Solve similar examples in same mods as homework examples.	Solve similar examples in mods not seen previously.	Solve examples which combine computational procedures.

Figure 2. Test plan used to construct the eight subtests of each form of the achievement test.

Although the nature of instruction was the independent variable in this study, other variables considered salient were one's verbal aptitude, one's mathematics aptitude, and one's previous mathematics background. Scholastic Aptitude Test scores were used as measures of the verbal and mathematics aptitudes. An intermediate algebra test was determined to be a related background measure, since computational procedures included in the instruction were of an

algebraic nature.

Procedure

The sample consisted of ninety-nine college students enrolled in a freshman level mathematics course in which modular arithmetic is usually included. There were three classes involved. Each class was randomly divided into two instructional groups. For four consecutive class periods each group watched a series of video-taped lessons taught by the investigator. Corresponding lessons in the two series included the same topics. The variable was the instructional strategy used in the presentation. Assignments were given which related to the instruction the group received. The class period following instruction the initial achievement test was administered and about six weeks later the retention test was given with no forewarning to the students.

FINDINGS AND DISCUSSION

The findings of the experiment supported both parts of the research hypothesis stated earlier. First to consider was that for any topic in mathematics there are several possible kinds of learning outcomes. A factor analysis of the eight subtests of the initial achievement test revealed two factors. Factor 1 indicated a distinction between the subtests identified with the declarative knowledge dimension and the subtests identified with the procedural knowledge dimension, with some commonness shared among some of the subtests across these dimensions. The second factor isolated the subtest of the clock representations from all the other subtests indicating a uniqueness about it. This supports Gagné and White's (1978) assumption of an images memory structure. The factor analysis of the retention test yielded similar results.

The second part of the research hypothesis--that the particular kind of outcome a student acquires is significantly influenced by the nature of the instruction--can best be discussed in terms of Figure 3.

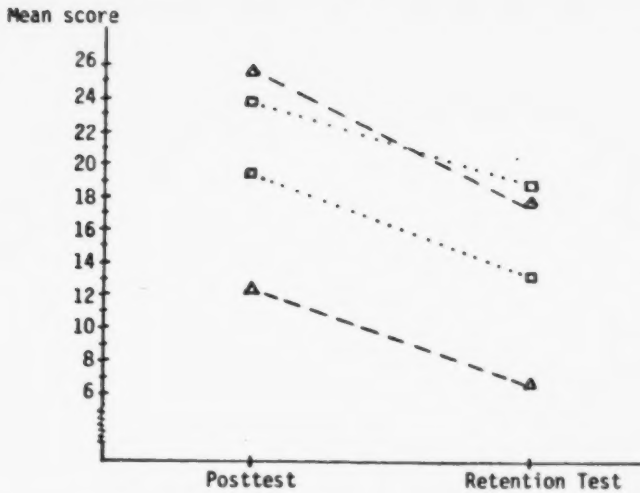


Figure 3. Mean scores of each instructional group on procedural and declarative subtests of posttest and retention test.

▲ Computational strategy - - - Procedural subtests
■ Verbal strategy ····· Declarative subtests

It can be observed that on the initial achievement test each instructional group scored highest on the dimension of learning outcomes expected. That is, the Computational strategy group yielded a higher mean score on the procedural subtests than the Verbal strategy group and the Verbal strategy group scored higher on the declarative subtests than the Computational strategy group. However, on the retention test the Verbal strategy group had a higher mean score on both sets of subtests than did the Computational strategy group. Along the declarative knowledge dimension the loss of learning was approximately the same for the two instructional groups, but along the procedural knowledge dimension, the loss was greater for the Computational strategy group.

In terms of Skemp's (1976) discussion of relational and instrumental understanding, these results are in agreement with some of his hypotheses. Instruction whose primary goal is instrumental understanding does develop greater computational proficiency for immediate use, but instruction stressing relational understanding provides for better long-term retention in the procedural outcomes.

Of the concomitant variables, mathematics aptitude was the only one correlated to both types of learning outcomes. Thus all further analysis of the data included it as the only covariate.

In summary the results of this experiment provided evidence to support the following hypotheses:

1. Procedural and declarative knowledge appear to be two distinguishable dimensions of learning outcomes.
2. The nature of the instruction has a significant effect on the type of learning outcomes achieved when mathematics aptitude is controlled as a covariate, and furthermore, the type of learning outcomes achieved is directly related to the instructional emphasis.
3. Instruction emphasizing meaningful mathematical relationships provides for greater long-term retention of procedural knowledge than instruction stressing computational competency.

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YOUNG WOMEN, CONSTRUCTIVISM, AND THE LEARNING OF MATHEMATICS

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RESUME

Vers la fin de leur adolescence, les jeunes filles obtiennent de moins bonnes notes aux tests mesurant la compétence mathématique. Toutefois, il est très bien connu que la plupart des étudiants éprouvent de la difficulté dans l'apprentissage des mathématiques. S'appuyant sur une philosophie constructiviste et sur les résultats de recherches touchant les processus cognitifs en jeu dans l'apprentissage des mathématiques, le programme "SummerMath" du collège Mount Holyoke a été conçu pour enseigner à ces jeunes filles des stratégies plus efficaces d'apprentissage des mathématiques. Cette tentative a donné lieu à l'élaboration d'un modèle innovateur pour promouvoir l'implication de la femme en mathématiques. Des facteurs tels que la confiance en soi, le goût du risque, l'indépendance et l'obstination, ont modifié notre vision de la philosophie constructiviste, et ont fait ressortir le besoin de trouver, dans cette perspective, une meilleure explication de la construction sociale.

The most recent results of NAEP confirm the findings of many early studies showing that although in early adolescence young women perform equally as well on achievement tests as young men, by the end of high school young men outperform young women in all tasks, especially those requiring understanding and applications--the higher level skills. When faced with findings of consistent differences in achievement between males and females, one needs to pursue two areas of inquiry:

- (1) How meaningful are the differences in relation to the performance of the group as a whole?
- (2) If the differences do indeed inhibit or restrict the access of the young women to power and resources, can one create an environment in which young women can learn mathematics in a way which promotes and encourages their equal participation?

I will discuss the first question briefly and spend a more significant amount of time suggesting the outlines of a response to the second question. Although researchers have consistently reported the findings of sex differences, those differences need to be set in the context of the widespread difficulties experienced by all students learning mathematics. Through clinical interviewing, educational researchers have revealed that often even when students perform adequately on achievement tests they hold complicated and consistent misconceptions about the mathematical ideas. These researchers report overreliance on memorization, lack of integration in students' concepts, and a tendency to blindly manipulate symbols, with little understanding. Thus, it is important to emphasize that while statistically significant differences in achievement between young men and women exist, the overall performance in mathematics is far from satisfactory for both sexes.

The achievement differences alone may not point to any significant denial of opportunity for young women. However, in addition to this data, the fact of the underrepresentation of women in mathematics-related fields does merit serious concern. If young women are being filtered out of careers commanding high status, high pay, and

weighty decision-making capabilities, then one needs to search for models of mathematics instruction which counter this trend. Such a model would take into account the factors hypothesized to influence female participation in mathematics, would utilize the considerable findings about the cognitive processes involved in mathematics learning, and would consider the research on the social configurations which promote female participation.

The SummerMath program at Mount Holyoke College is designed with such a model in mind. SummerMath is a six-week, intensive program designed to address the national concerns about the inadequate participation of young women in mathematics. The 104 students enrolled in the program are academically capable but find mathematics a hurdle, relying as a result on memorization and imitation of textual examples. Many of these students describe themselves as math-anxious. By teaching these young women problem-solving strategies, introducing them to computers, and demonstrating the application of mathematics to the real world, the staff aids students in gaining competence and confidence in mathematics. The program is based on the research on cognitive processes as interpreted through a constructivist perspective. To describe this philosophy, I will describe its most fundamental tenet, as stated by Glasersfeld (1974):

A person's representation of the environment, his/her knowledge of the world, is under all circumstances the result of his/her own cognitive activity. (p. 22)

Within the program, five major tenets followed from this basic assumption:

- (1) Students, using their previous representations and knowledge, will construct a meaning for what they are learning; and the development of more elaborate and unifying constructions should be encouraged.
- (2) Our capacity for reflection upon our previous representations and the construction of these representations is essential for developing increasingly adequate constructions and evaluating the

success of those constructions. This capacity needs more direct attention in instruction (Confrey, 1981).

(3) The current portrayal of the subject matter of mathematics in classrooms is inadequate to promote the development of effective personal constructions. It is portrayed as rigid, absolute, and external, not as a human construction. For instance, problems are not defined as roadblocks to where a student wants to be (Confrey, 1981).

(4) Teaching, according to constructivism, requires the nurturing and promotion of more adequate and effective constructions. Although teachers cannot provide constructions for students, they can select what they hope and evaluate to be likely problems to promote the refinement, unification, or elaboration of students' constructions. Teachers, according to this model, are very active; constantly anticipating, interpreting, and testing out their hypotheses about students' constructions by questioning and providing potential problems.

(5) An individual will also construct representations of the beliefs and purposes of the community of others in which s/he resides. Classrooms and schools are such communities, and within these the sharing of personal constructions is undertaken. One can only see another's construction through one's own. The implications of such a tenet for developing an alternative model of mathematics which nurtures participation by young women are substantial. Because these issues have grown to influence my conceptualization of SummerMath continuously, I will discuss this tenet in more detail later.

The implications of a constructivist philosophy are felt throughout the curriculum and within the instructional strategies. Within the program, instructors use student "errors" as opportunities to explore what beliefs are held by the student about the concept. The emphasis is always on what a student believes and why she believes it. Questions are answered with questions, in the belief that if a student can pose and phrase a salient question she is well on her way to solving the problem. We emphasize the importance of having

alternative strategies for solving a problem, including using trial and error, arithmetically, algebraically; and whenever possible we use a variety of representations. By promoting flexibility, we believe that we can decrease the students' reliance on memorization and unify their concepts.

A heavy emphasis on the use of language is pervasive in the program. Students work in pairs and in small groups as they solve problems. Learning to speak mathematics is essential to develop and promote the importance of precision in mathematics. We also find that solving problems in pairs encourages the kind of self-reflection we believe leads to more effective conceptual development.

The use of LOGO in the computer classes is also an important part of the program. The beauty of LOGO is its creation of a slice of the world that students can manipulate and explore and which is rich in potential for mathematical insights. Using LOGO contributes tremendously to the constructivist goals for the program; students set their own goals, learn to think analytically, and obtain immediate feedback on their endeavors. They learn to expect to debug programs, and that encourages them to be more persistent in their pursuit of mathematical ideas. LOGO is an example of a microworld that nurtures a student's sense of empowerment as she selects the problems to explore and sets the standards for success.

The strategies for teaching and learning mathematics in SummerMath evolve neatly from the research on mathematics learning. The results of using such methods with a community of young women include evaluations by them which repeatedly speak of their increased persistence in solving problems, a sense of the accessibility and humanity of mathematics, a greater tolerance of ambiguity, and increased levels of engagement.

Increasingly I have found it inadequate to focus solely on cognitive processes or on contextual factors influencing the participation of young women. Certain other factors have emerged persistently to tug

at the edges of my consciousness. In thinking about mathematics anxiety¹ these issues are raised into focus.

In personal interviews and in a two-week workshop on mathematics anxiety it became apparent that numerous other social factors contribute to the difficulties experienced by young women in learning mathematics. These include a real aversion to the extent to which they see mathematics classrooms as a place of public evaluation. Our students repeatedly said that they would not ask questions if they were confused, for fear of being judged stupid by the other students or by teachers. This hesitancy to ask questions was increased repeatedly as they relied so heavily on memorization to masquerade as understanding that they felt they were participating in a rude hoax. Any question might reveal the depth of their ignorance. They felt totally alone in their inadequacy.

What they saw as valuable in the methods of SummerMath was the replacement of competition with cooperation. They expressed delight in seeing the pace of content coverage slowed as understanding and flexibility were stressed. And, finally what we saw as a dramatic change in them was an increased sense of independence, self-responsibility, and autonomy. They began to set their own standards for justification and began to develop confidence in their own abilities.

It is the issue of confidence which I found to be the most revealing of the inadequacies of my prior conceptions of learning. It seems to me that with many bright, talented young women we fail to get any measure of what their capabilities in mathematics are because they lack the confidence to solve problems vigorously. I believe an adequate theory of mathematics learning must account for this factor.

For constructivism to explain the factors of confidence, autonomy, willingness to risk, and self-responsibility, it is necessary to examine in more detail how the fifth tenet on constructions of

community-wide standards functions. Although my thinking on this is only tentative, I believe that a young woman forms a set of beliefs about her own level of understanding in relation to her beliefs about the level of understanding of the group. The adequacy of one's own construction is not solely a function of its power to effectively explain one's outstanding problems but as a way of locating oneself in relation to others.

It is the development of an interpretation of constructivism which explains how expectations and feedback from a group contribute positively or negatively to the promotion of more adequate constructions which defines my task in the months to come.

Notes:

1. The conceptualization of math anxiety has been undertaken jointly by Dr. Patricia Romney, Dr. Joan F. Mundy, and myself.

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AN EXAMINATION OF THE UNDERSTANDING LEVELS OF JOHN WILSON'S
ACTIVITY TYPE CYCLE: A CASE STUDY

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RÉSUMÉ

John Wilson a développé un modèle pouvant servir de guide dans la mise au point d'activités d'enseignement. Ce modèle dit "Activity Type Cycle (ATC)" comprenait des classes d'activités où l'accent était mis sur les différents niveaux de compréhension requis pour la consolidation des connaissances et la résolution de problème. Quoique ce modèle ait été appliqué à l'enseignement en classe, l'intention était ici d'examiner en détail deux des niveaux du modèle (l'abstraction et la schématisation) lorsqu'utilisé dans un contexte clinique. Un certain éclairage a été apporté par l'analyse des travaux de deux enfants du "Mathematics Education Diagnostic and Instructional Centre" de l'Université de Colombie Britannique. Le second niveau de compréhension, la schématisation, a fait l'objet d'une attention particulière pour faire ressortir l'importance de promouvoir les activités de nature à développer l'habileté des enfants à organiser leurs pensées et observations.

Many mathematics educators have developed models for guiding instruction. For example, Steffé proposed a model which lists three phases of instruction: exploratory phase, abstractions and representation phase and formalization and interpretation phase (Davis, Mahaffey, McKillop and Steffé, 1974). Dienes stated that concept formation develops in three stages: a "play", a "structured" and a "fixing" stage (Dienes, 1967). The characteristic feature of most of the models is that emphasis is given to providing "understanding" activities to children before those which encourage children to master the mathematical content. It is the purpose of this paper to take a closer look at these understanding activities.

THE ACTIVITY TYPE CYCLE

John Wilson presented a model for instruction that contains six types of activities (Ashlock, Johnson, Wilson and Jones, 1983). An outline of the model, known as the Activity Type Cycle (ATC), is given in Figure 1. The core of the model is the diagnosing activity type and is a reflection of Wilson's belief in the diagnostic-prescriptive nature of the teaching process. Initiating is perhaps best understood as a discovery stage where children are encouraged to improvise solutions to problems based on what they already know. During abstracting, instruction is typically more structured and is designed to help children focus on specific attributes of a single concept. Schematizing activities are intended to develop children's ability to discover and examine interrelationships among concepts or skills. Note that abstracting typically involves one concept only (such as $4+2=6$) whereas schematizing involves more than one (such as if $4+2=6$, then $4+3=7$). Consolidating activities help children to master the new concepts or skills and generally require the use of games and drills. Transferring activities are those which permit children to apply the concepts and skills to novel situations and represent the problem solving stage of the instructional model.

It is recognized that this overview of the model is too brief to adequately illustrate all of its features yet it is important to

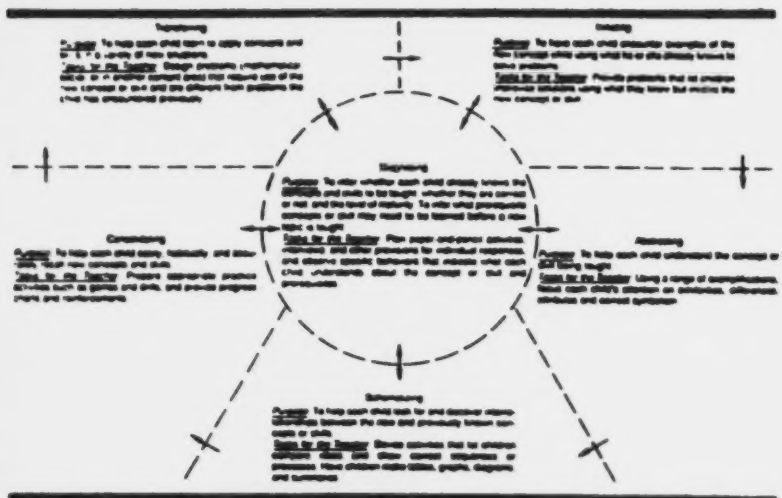


Figure 1. Activity Type Cycle: Purpose of Activities and Tasks for the Teacher (from Ashlock, Johnson, Wilson and Jones, 1983)

understand where the abstracting and schematizing activities are located in relation to the other activity types. Before considering abstracting and schematizing in some detail, one point should be noted. The activity types are not always as distinct as they appear. When one asks a child 'What do I have to add to 4 to get 7?' it may not be clear whether that is a problem solving example (transferring) or is an introduction to the missing addend concept of subtraction (initiating).

ABSTRACTING AND SCHEMATIZING

When the Activity Type Cycle was first developed it contained only five activity types: diagnosing still as the core and initiating, understanding, consolidating and transferring. Shortly thereafter Wilson recognized that the understanding activity type was so broad that planning for instruction would be better served if it was divided into two parts - abstracting and schematizing.

During abstracting activities children are introduced to definitions and symbols. They should learn to write equations, to perform algorithms, to understand the meaning of concepts or operations. Pedagogical procedures include using concrete materials and working toward a symbolic level of understanding. For teaching the basic whole number facts typical activities might involve counters, Cuisenaire rods, a math balance and number lines. For place value the activities might utilize bundled sticks, Dienes' blocks, pocket charts and expanded notation.

With schematizing activities many of the same materials, diagrams and models would be used again. In this case, the concern is to demonstrate interrelationships among various concepts. When appropriate, children are encouraged to formulate generalizations. For example, a child could be asked to generate all the basic addition facts that have a sum of six using counters, a math balance or a number line. The child could then be directed to place these facts in some suitable order, say $6+0=6$, $5+1=6$, $4+2=6$, and so on. By returning to the material or model, the child could then be shown that by removing one from one addend and adding it to the other addend, the sum remains the same. Finally, the child would be asked to verbalize what he or she thinks is happening. This might be considered the first step in developing the compensation in addition property.

At the Mathematics Education Diagnostic and Instructional Centre (MEDIC) clinic at the University of British Columbia, this division of "understanding" into abstracting and schematizing appears to have been particularly useful when working with the clinic tutors. The division helps focus attention on the need for schemata. DeBono (1983) defined perception as "the way our minds make sense of the world around us" (p. 704). "Perception takes place in an "active" information system. Such systems allow experience to organize itself into immensely useful patterns, ..." (p. 705). This need to allow incoming experience to organize itself into patterns is crucial. The separation into abstracting and schematizing

facilitates concentration on activities that encourage children to organize, to attend to interrelationships among various concepts.

It could be argued that the division of understanding into two levels is arbitrary. It might make more sense to think of an understanding level as a progression along a continuum from a single concept to that of highly complex sets of interrelationships among concepts. Some evidence for this argument might result when one considers, for example, the commutative property of addition. Is this a single concept - that the order in which two whole numbers are added has no effect on their sum? Or, is this a set of concepts associated with numerous observations of applications of the property? Abstracting or schematizing?

Although further evidence might be provided, it is not the intention of this article to suggest modifications to the ATC. The comparative simplicity of having only two levels of understanding is helpful. Yet having the separation into two levels, rather than just one level, is also helpful in that some attention is given to the pedagogical need for schematizing activities.

TWO CASE STUDIES

Portions of the work of two children who attended the UBC MEDIC clinic were selected because they illustrate good applications of schematizing activities.

David, age 12 and in grade seven, was referred to the clinic by his parents. Testing showed that David was of at least average intelligence. He appeared to have particular strengths in abstract reasoning ability and proficiency in visual-spatial relationships. His weaknesses were in the areas of written language and numerical reasoning. His mother reported that among other things they had been working on his "basic facts" for over five years with no apparent success. Detailed mathematics testing indicated that David had some place value understanding provided that the number did not

contain a zero (6001 was read as 601). He could recall very few basic addition and subtraction facts and confused the role of zero ($6+0=0$). He did not apply commutativity. David could not add nor subtract whole numbers with regrouping.

David attended the clinic for 14 bi-weekly sessions over an eight week period. Of each 75 minute session approximately 25 minutes was devoted to "learning the basic facts". Initially, David worked on the addition facts to 10, then to 18, using interlocking Unifix cubes and the number line. He learned to record what he manipulated or observed and could illustrate a written example with the materials (abstracting level). Then David started working on the commutative and associative properties utilizing the same materials. This was followed by an introduction to several learning strategies (Davis, 1978; Thornton, 1978). Emphasis started with the +1, +2, +0 and "doubles" facts. These were then used to demonstrate, for example, that if $2+2=4$, then $2+3$ could be found by adding one to the four. David was asked to compare flashcards such as $4+4=8$ and $4+5=9$. Are they the same in some way? How? Are they different? How? Later he was given a card, say $5+5=10$, and asked to choose a card that "was close to it in some way" ($4+5=9$ or $5+6=11$). Eventually this led to the construction of the entire addition table in stages from $0+0$ to $5+5$, to $7+7$, and to $9+9$ (schematizing level) (Lazevick, 1981).

When David left the clinic he could recall with speed and accuracy all facts to $6+6$. He generally attempted to answer those facts he couldn't recall by thinking of "a close one". His attitude toward mathematics improved immensely. What he hadn't been able to accomplish in five or six years he had mastered in seven weeks.

Jeffrey, age 8 and in grade 2, was also referred to the clinic by his parents. Unlike David, who tended to be quite, Jeffrey was active and often difficult to manage. Although he scored in the slightly below average range on the WISC-R, this may have been due to attentional and behavioral problems. Specific mathematics

testing indicated that Jeffrey had very poor understanding of place value concepts although he could read and write two digit numbers. He could recall basic addition facts to five but generally relied on finger counting to verify accuracy. He could also recall a few subtraction facts but could not relate unknown subtraction facts to known addition ones ($7-3=?$ when $4+3=7$).

Jeffrey spent two semesters on a bi-weekly basis for a total of 36 sessions at the UBC MEDIC clinic. The 75 minute sessions were typically divided into three 25 minute periods. One period focused on place value, the second on basic facts and the third on some project or game. The place value activities were eventually extended to include addition and subtraction (without and with regrouping). Extensive use was made of bundled sticks, a pocket chart and base-ten stamps. Some use of base-ten blocks was attempted but, as a result of some unexplained resistance on Jeffrey's part, they were not emphasized. Initially, Jeffrey was asked to construct two and three digit numbers (such as 27, 83, 465, 240, ...) with the bundled sticks and base-ten blocks (abstracting level). He was then asked to prepare in sets 56 and 65; 123, 321, 312; 24, 240, 204; and so on. In each case he was asked to explain what was the same about the numbers? What was different? Followup activities included asking Jeffrey to add one, then 10, then 100, ... (concretely then symbolically) to a certain number. Finally, he was asked to complete sequences of the form 61, 71, 81, __, __; 778, 878, __, __; and 524, 514, __, __ (schematizing level). At the end of the second semester Jeffrey was adding and subtracting two and three digit numbers (with regrouping) confidently and accurately.

One is hesitant to attribute the progress of a child exclusively to clinical intervention. Obviously many factors such as parent participation, regular classroom learning and individual attention during tutoring all contribute to the improvement. However, when youngsters such as David and Jeffrey begin to achieve some measure of success after a lengthy period of failure there is a need to examine any special or atypical component of a particular instructional plan.

It is suggested that with these two children the focus on schematizing activities was instrumental in their progress.

CONCLUSION

This paper was written to emphasize the need to indicate much more than just "we must teach for understanding before we drill". "Understanding" needs to be developed from its lowest levels of being able to illustrate a concept with only one material to levels of sophistication which include seeing complex interrelationships among associated concepts. John Wilson's Activity Type Cycle was offered as a helpful instructional model in that the model contains two levels of understanding. The second level, schematizing, was given specific consideration to stress the importance of promoting activities that develop children's abilities to organize their observations and thoughts. Efficient storage systems should result in better retrieval mechanisms.

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COGNITIVE DIAGNOSIS OF CHILDREN'S ARITHMETIC

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RESUME

Trois approches différentes de diagnostic de la pensée mathématique sont décrites. Un type de diagnostic a une fonction normative - la comparaison d'individus sur un trait quelconque - et est habituellement réalisé à l'aide de tests standardisés. Les tests de cette nature peuvent être améliorés en leur donnant une base théorique. Un nouveau test diagnostique de ce type est décrit. Un second objectif du diagnostic est la mesure précise des processus cognitifs. Quelques succès de cet ordre peuvent être obtenus par les tests standards. Un nouveau test de ce type est décrit. Cependant, le second type de diagnostic peut être mieux réalisé au moyen de l'entrevue clinique. L'essence de l'entrevue clinique est décrite. Un troisième objectif du diagnostic est de tracer un portrait holistique de l'individu. Ceci doit être tiré de techniques d'évaluation variées quoique nous n'ayons pas, en ce moment, les théories adéquates pour réaliser cet objectif.

The purpose of this paper is to show how the psychology of cognitive development, particularly recent research on children's mathematical thinking, leads to important applications to the problem of diagnosis in the school setting.

THE GOALS OF DIAGNOSIS

The essence of diagnosis is the attempt to develop a sound psychological theory of the child's functioning in academic situations. Traditionally, diagnosis has adopted a normative goal, involving a comparison of the child with peers on a cognitive attribute or dimension. The main question to be answered is: How does the individual compare with peers on key indices of achievement and ability? The main technique employed in answering this question is standardized testing. Normative diagnosis is lacking in that it fails to specify cognitive process, and instead focuses on comparisons among children on imprecisely defined mental traits. To overcome this limitation, we must supplement the normative approach by two other types of diagnosis having distinctly cognitive goals.

The first additional goal for diagnosis may be termed the cognitive purpose: it involves the discovery and measurement of specific cognitive processes. The main question to be answered is: what cognitive activities generate the observed behavior, namely difficulty in school arithmetic? The aim is not to compare children but to characterize specific cognitive processes with as much precision as possible. Some success in answering such questions may be achieved by standardized testing, although, as we shall see below, a more powerful method is the clinical interview.

Another purpose of diagnosis is the integrative. This involves a description of the configuration of cognitive and other psychological processes within the individual. The main question to be answered is: what is the nature of the unique constellation or Gestalt of cognitive processes characterizing the individual? To answer a question like this, a variety of methodologies need to be employed, ranging from standardized testing to naturalistic observation.

THE NORMATIVE PURPOSE

Recent cognitive research makes a vital contribution to normative diagnosis by suggesting key aspects of mathematical thinking which should be measured. Cognitive theory is also useful in identifying certain weaknesses in the methods usually employed in normative diagnosis.

Several normative tests (e.g., the KeyMath, [Connelly, Nachtman, and Pritchett, 1976] have attempted to provide measures of general mathematical ability like "computational" skill. Normative tests of this type are of limited value since the theories on which they are based are vague. Notions like "computational skill" mainly point the way to areas of thinking which deserve closer theoretical attention and empirical exploration. Modern cognitive research does not conceptualize mathematical thinking in terms of general traits. Instead, it attempts to describe in detail the cognitive processes underlying mathematical performance. Also, trait tests are not particularly useful from a practical point of view. To know that the child is weak on "general mathematical ability" does little more than remind the teacher of what is already well known, namely that the child needs help in mathematics.

Fortunately, recent cognitive research provides conceptions of mathematical thinking which permit the construction of normative tests comparing children on theoretically important aspects of mathematical knowledge. For example, the Test of Early Mathematics Ability (TEMA), which we have recently developed (Ginsburg & Baroody, 1983), compares children not only on what they have learned in school (like alignment procedures for column addition) but on their "unsuspected strengths," the informal procedures developed outside the context of schooling. For example, the test contains items dealing with the addition of concrete objects, the enumeration of dots, mental addition and subtraction, and the mental number line. The TEMA, which is given in a standardized fashion and has been normed on a large sample, yields measures of specific formal and informal concepts and procedures, not general mental traits. Such a diagnosis is intended to provide

a rounded view of the child experiencing difficulties in school arithmetic.

THE COGNITIVE PURPOSE

Having learned that a child is in the 20th percentile in general numerical ability, the diagnostician next inquires into specific cognitive processes underlying the child's work. One way of approaching this issue is by means of a standardized test focused on specific cognitive processes like those underlying errors of written calculation. Recent research shows that errors are often caused by underlying "bugs." These are procedures which are fundamentally defective and in certain situations lead to regular, predictable errors. For example, in subtraction, one common bug is when the child operates according to his own rule, "Always subtract the smaller from the larger."

Bugs can be measured by standardized procedures. The basic logic involves presenting the child with written calculational problems carefully selected so as to reveal the operation of key bugs. For example, suppose one wishes to identify the error strategy "Always subtract the smaller from the larger." To measure this bug, it is necessary to present the child with problems of the form $82 - 28$, where the subtrahend (28) has a number in the units position (8) which is larger than the units number (2) in the minuend (82). On problems like these, use of the bug results in a clearly predictable and distinctive answer--66 in this case. This answer is most likely to have been generated by the bug in question. If several carefully selected problems of this type are employed, then the child's calculated answers to the written problem provide useful evidence concerning the underlying strategy.

While useful in measuring clearly defined cognitive processes, a standard test of this type suffers from limitations which can be overcome by more flexible procedures of testing. For example, a difficulty arises when the child employs a bug that the test is not designed to measure. It is obviously impossible for a standardized

diagnostic test, which must be administered in about 30 minutes, to measure the literally thousands of distinguishable bugs that have been shown to exist (Brown & Burton, 1978). Hence, on a test of this type, those answers generated by bugs which were not intended to be measured must remain uninterpretable.

One solution to these difficulties is to supplement the standard test format with a kind of talking aloud procedure in which the child is asked to "say everything that comes into your head" while solving the problem. The pioneers in this area were Buswell and John (1926), who used carefully selected problems and children's verbalizations to measure the "habits" of arithmetic. Recently, we have developed the Diagnostic Test of Arithmetic Strategies (Ginsburg & Mathews, 1983) which also uses both specially designed problems and verbalizations to measure the most frequent bugs in addition, subtraction, multiplication, and division.

Despite their virtues, standard tests like these are limited because they do not permit the tester to explore possibly interesting responses. If the tester adheres to standardized testing procedures, he may learn something from an unusual written response and an intriguing verbalization, but is not free to explore matters in a productive fashion.

Fortunately, Piaget (1929) long ago developed a research procedure, the clinical interview method, designed to overcome limitations like these. The method is as useful for diagnosis as for research. Implicit in the clinical interview method are three distinguishable purposes: discovering cognitive activities; specifying them with precision, and evaluating levels of competence.

Piaget's first aim, discovery, is the least controversial of the three. One basic task for diagnosis is to discover the cognitive processes actually used by children in a variety of contexts. To accomplish this, Piaget designed the clinical interview method, which he described as an unstructured procedure intended to give the child

a chance to display his "natural inclination." When discovery is the aim, the examiner employs an open-ended task, asks questions in a manner contingent on the child's previous responses, and requests a good deal of reflection on the part of the child. The interviewer attempts to avoid putting words in the child's mouth or to suggest answers--these are both clear dangers of the clinical interview method. The clinical method is intended to draw from the child, in an unbiased manner, rich material allowing the discovery of unsuspected cognitive processes. If the child in a diagnostic session displays an unusual or unexpected answer, what could be more useful for the examiner than to follow up on it in a manner which is deliberately unstandardized--that is, which is contingent on the child's response?

The aim of specification may also be served by the clinical interview method. Once interesting intellectual phenomena have been discovered, the cognitive processes underlying them need to be measured and described in detail. If, for example, we learn from exploratory work that a child adds by counting, we next need to describe the strategy in detail. When specification is the aim, the interview employs a technique which is usually not recognized as characteristic of it, namely the experimental method. If, for example, the interviewer feels that the child experiences difficulty with the materials employed in the task, she may systematically vary them while holding language constant. If a method like this were undertaken with a large enough number of subjects and an appropriate counterbalancing procedure, it would be an example of traditional experimental design, necessary for deciding among alternative hypotheses. In fact, one may describe clinical interviewing for the purpose of specification as the flexible and extremely demanding application of experimental methodology in a manner contingent on individual reactions.

A third aim is the assessment of competence. In the diagnostic situation, it is usually all too well known that the child in question is performing poorly. The real question refers to competence: What is the best he can do? Determining competence depends in part on insuring that the child is properly motivated to work on key mathematical

problems. And, to motivate a child, one needs to tailor the task to his individual needs by means of the flexible construction and presentation of tasks. For whatever reasons, a child may find one form of a task boring or overly difficult, but not another. To overcome this, the examiner must create new tasks, often on the spot, in response to the needs of individual children. The clinical interview method can adjust the method of presentation to promote motivation in the individual; the very logic of standardized testing prevents it from employing such flexibility.

THE INTEGRATIVE PURPOSE

A third aim of diagnosis is to obtain an integrated portrait of the individual. The educator and clinician are always faced with the necessity of understanding "the whole child"--for example, a child who not only performs poorly in school arithmetic, but also feels anxiety about it, believes that he is a "dummy," dislikes school, is not doing very well in reading either, withdraws in the face of criticism, finds it hard to concentrate when difficult problems are presented, is dependent on the teacher for problem solving suggestions, is afraid to show his grades to his parents, lives in a family where the parents have a strong fear and dislike of mathematics in particular and schooling in general, and so on. A comprehensive understanding of the child must include consideration of factors like these. Yet, contemporary cognitive research has focused mainly on what might be called the "technical" aspects of mathematical cognition--e.g., bugs. Only a few writers have attempted to suggest alternative approaches. For example, Papert (1980) stresses that the affective component in learning mathematics is as crucial as knowledge of the subject matter itself. Despite such excellent efforts, a good deal remains to be done to develop new theoretical models and appropriate methods for understanding the whole child.

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MENTAL STRUCTURAL IMAGES
CHARACTERIZING VAN HIELE LEVELS OF THINKING

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RÉSUMÉ

L'étude de la notion piagétienne de groupement dans le contexte de la didactique de la mathématique a montré sa pertinence et son applicabilité dans l'analyse a priori, mathématique-épistémologique de contextes conceptuels ou de situations de problèmes de l'enseignement de la mathématique. Récemment, des tentatives ont été entreprises en vue de vérifier et de décrire la formation et le développement de groupements chez l'individu (LUNKENBEIN/ALLARD/GOUPILLE, 1983 a, b). Ces expériences semblent indiquer trois types de groupements (groupements infralogiques, groupement (logique) des classes d'une partition et groupements logiques d'inclusion) comme étant de première importance pour décrire le cheminement d'apprentissage et des étapes de progression dans l'élaboration d'un contexte conceptuel par l'individu. Le caractère de ces groupements et les réactions observables des apprenants suggèrent un rapprochement de ces structures de pensée avec les trois premiers niveaux de pensée formulés par van Hiele, plus particulièrement avec l'idée des réseaux de relation sous-jacents aux divers niveaux. Nous considérons alors ces trois types de groupements comme la description de structures cognitives sous-jacentes aux trois premiers niveaux de pensée selon van Hiele. Nous les concevons en tant qu'images mentales structurées d'un contexte conceptuel, construites progressivement l'une sur l'autre et représentant des perspectives graduées du contexte en question.

THE STUDY OF GROUPINGS IN MATHEMATICS EDUCATION

The piagetian concept of grouping is of particular interest to the mathematics educator who considers learning of mathematics as a constructive process of model building originating from the student's experience in a given context and aiming at the mastery of such context in terms of satisfactory explanations, predictions and coordinations. Groupings may be thought of as results of cognitive activity implied in such model building processes and thus describing the model constructed or being constructed (WITTMANN, 1982, pp. 277-280). The study of groupings in mathematics education therefore aims at the description of products of cognitive activity in a given conceptual context or problem situation as well as the possible development of such products according to the subject's experience or intentions.

First efforts in this domain were concentrated on a satisfactory formulation of the concept of grouping. Starting from Piaget's original formulation (PIAGET, 1967) a series of papers by psychologists and mathematics educators led up to a very general and formal description by E. WITTMANN (1975) in terms of mathematical categories. As the work on the formulation of the concept progressed, applications of the concept of grouping in mathematics education were described. (LUNKENBEIN, 1981; WITTMANN, 1982). Such applications were the result of mathematical and epistemological analyses of conceptual contexts or problem situations taken from mathematics teaching and did not attempt to verify or to describe the formation of groupings by individual learners. It is only recently that such attempts have been undertaken by a group of mathematics educators at the Université de Sherbrooke, the global results of which suggest the conception of groupings as underlying mental structures of levels of thinking as will be outlined in this paper.

DESCRIPTION AND EXAMPLES OF GROUPINGS

For the purpose of this paper we choose to describe the concept rather informally and to illustrate groupings by some examples (for the formal definition refer to WITTMANN, 1975).

Essentially, a grouping consists of

- (1) "static" elements, called states or objects and which represent instances of a given conceptual context or problem situation;
- (2) "dynamic" elements, called operations or transformations that act on the states or objects and relate them to each other;
- (3) properties of or relations between states or objects brought out as the effect of the operations on the states. (WITTMANN, 1982, p. 285).

We illustrate this description by three examples taken from the conceptual context of quadrilaterals.

First, we consider a grouping that might emerge directly from concrete activities with quadrilateral shapes. Such shapes may be given as cardboard cut-outs; through cutting or pasting, the various quadrilateral shapes can be transformed into each other. Such concrete transformations may be composed (at least in the mind) and may lead to a mental structural image of the context which could look somewhat like this.

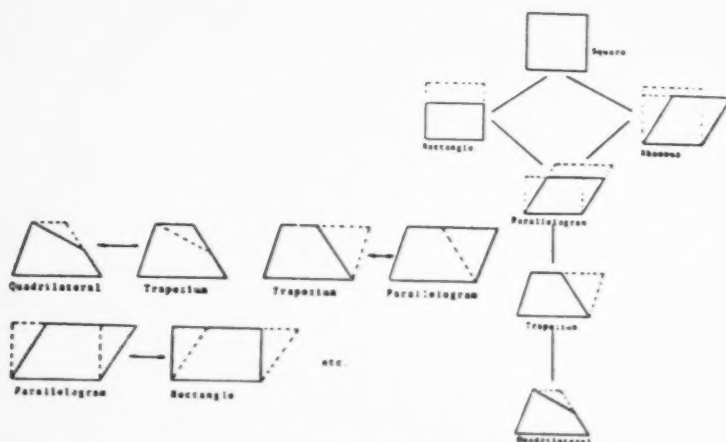


Figure 1: Relating quadrilateral shapes through spatial operations

In this grouping, the objects are particular quadrilateral (cardboard) shapes, recognized by global perception and identified by particular, rather accidental names. The operations are the cutting or pasting in various ways of given objects in order to obtain other given or new objects. The effects of such operations on the objects are that they make sides "alike" (parallel), "corners right" or sides "equal" etc. By this way, relevant properties of quadrilateral shapes are brought out, formulated and clarified.

Second, we consider an activity with quadrilateral shapes (that may be given in concrete form) where properties like parallelism of sides, equiangularity and equilaterality are well known and where these shapes are being classified according to such properties. Simultaneous application of the given properties leads to the classification shown in figure 2. All rectangles are put together in a collection of shapes that have two pairs of parallel sides and are equiangular; all squares are united in a class where all shapes have two pairs of parallel sides, are equiangular and equilateral, etc. Thus, shapes are now classes of objects and can be described by their properties.

By adding or retrieving properties, shapes can be transformed into each other, which may result in a mental structural image of the conceptual context as follows.

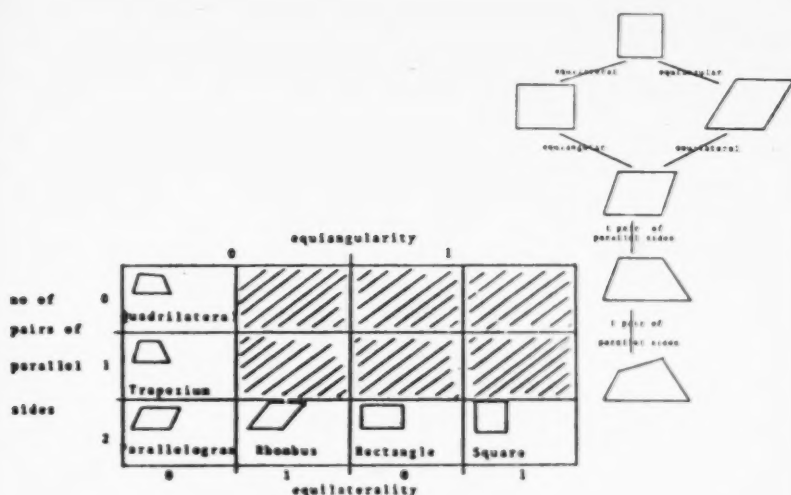


Figure 2: Classifying and relating quadrilaterals by adding or retrieving properties

In such a grouping, the states are classes of shapes described by relevant properties. These states are compared through the presence or absence of certain properties and can be transformed into each other by the adding or retrieving of properties. The effect of such operations is to bring out the relative importance or significance of such properties in the context of quadrilaterals. All equilateral quadrilaterals have two pairs of parallel sides and, equally, there are no equiangular quadrilaterals that are not parallelograms. In this manner, relations between properties of quadrilaterals are conceived, formulated and developed.

This leads to a third type of grouping, based on categorizing activities of quadrilaterals according to defining properties. Quadrilaterals Q, defined as four sided polygons, include trapezia T, which are defined by the additional property of having (at least) one pair of parallel sides. Rectangles R can be included in the category of parallelograms, if we can show that the property of equiangularity in the context of quadrilaterals implies the property of having two pairs of parallel sides. In pursuing such reasoning the following mental structural image may emerge.

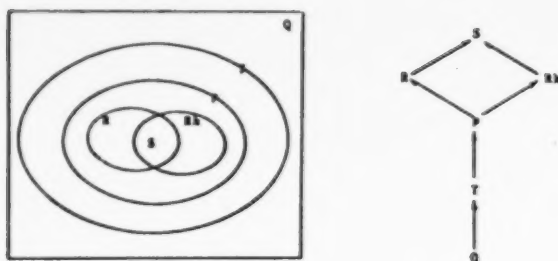


Figure 3: Relating quadrilaterals through logical inclusion

In such a grouping, the states are geometrical concepts given by their defining properties. Such states are related to each other by a hierarchical ordering based on the logical implication of corresponding properties. States are transformed into each other by the conception of inclusions of categories on the bases of logical deduction. The effects of such logical operations are that they establish a network of logical relations in the given conceptual context which can subsequently be organized into an axiomatic system describing formally the context in question.

TYPES OF GROUPINGS

The preceding examples not only illustrate groupings as mental structural images resulting from cognitive activity in a given context but also show, that such groupings can be essentially different by the nature of their states and operations and thus describe rather different outlooks on the same conceptual context. Three types of groupings are illustrated in these examples.

The first is infralogical in nature. Such groupings are characterized by the spatio-temporal aspect of their states and operations. They relate spatial figures to each other by decomposing such figures in constituent parts and by (re)composing them into an organized whole. The constitution of the whole depends essentially on the spatial contiguity of its parts and infralogical operations can be performed mentally only if the corresponding localization of the parts is performed in the mind. Infralogical groupings are the result of activities involving concrete material and actions and organize these elements spatially and visually.

In opposition to infralogical groupings, logical groupings deal with discrete elements, discriminate and unite them into collections, classify and relate them to each other in various ways. Two types of logical groupings are illustrated in the preceding examples. Groupings of the classes of a partition, illustrated in the second example, are constituted of the classes of a partition and relate such states through comparison of the presence or absence of the properties on which the partition is formed. Such groupings are the result of classifying activities according to given criteria and organize such classes in a descriptive way. Groupings of logical inclusion, illustrated in the third example, consist of categories of objects defined by properties and connected by logical operations or relations. They are the result of categorizing activities according to precise properties and the hierarchical ordering of such categories by the relation of inclusion which, in turn, is based on logical relations between defining properties. Thus, categories or concepts are defined and organized in a logical way.

The relation of infralogical to logical groupings is described by Wittmann as follows: "It is of paramount importance for the psychogenesis that, starting from senso-motoric patterns, the infralogical domain is first constituted. The logical operations and relations remain subordinated to the infralogical ones through a long period and need a lengthy process to free themselves from this domination " (WITTMANN, 1975, p. 59). Our own experiments confirm this relation insofar as all subjects having had no prior experience in a given context, displayed reactions to be situated in the infralogical domain and insofar as accession to a logical domain always required a rather lengthy learning process or specific teaching interventions (LUNKENBEIN/ALLARD/GOUPILLE, 1983, a, b). After such learning processes or specific interventions only manifestations of groupings of classes of a partition could be observed even when the interventions were designed to introduce logical inclusion. The latter thinking patterns could only be observed in subjects that had acquired some maturity with respect to the given context through ample experience and appropriate knowledge. So it seems that groupings of classes of a partition precede, in the logical domain, groupings of logical inclusion. The illustrated sequence of thinking patterns was observed not only in subjects of school age, but also in adults (inservice teacher training).

LEVELS OF THINKING AND GROUPINGS

The formulation of such types of groupings and the apparent description of thinking patterns through structural mental images of a given context suggest the comparison with levels of thinking, particularly with those described by Pierre-M. van Hiele (WIRSZUP, 1976). The essential idea of van Hiele's levels of thinking is the assumption that, in every scientific subject, it is possible to think and to reason at different levels and that such reasoning depends on a network of relations established in a given context. Such networks can be essentially different in nature and therefore cause different patterns of thinking which, in turn, are manifest through appropriate action or language (van HIELE, 1959). In the description of the levels of thinking, based on the idea of an underlying structure and expressed in terms of reactions, language expression and reasoning, van Hiele speaks rather vaguely of visual or spatial structure, of descriptive structure and of logical or theoretical structure respectively (van HIELE, 1959). Such rather vaguely indicated underlying structures seem to be more precisely described, for the first three levels of thinking, by the three types of groupings given before.

Infralogical groupings seem to describe precisely the underlying visual or spatial structures of van Hiele's first (visual) level of thinking. Subject's reactions in terms of manipulating, composing and decomposing, naming concrete objects and not being able to compare or to describe objects by characteristic properties are well explained by such infralogical structures. Groupings of classes of a partition seem to correspond to the underlying descriptive structures of van Hiele's second (descriptive) level of thinking. The conception of objects as bearers of properties, the comparison and description of objects by relevant properties as well as the absence of logical relations between such properties are pertinently suggested by groupings of classes of a partition. Finally, groupings of logical inclusion seem to correspond to the underlying logical structure of van Hiele's third (logical) level of thinking. The possibility of logical ordering of properties of objects, the motivation for and understanding of definitions, the emergence of deductive reasoning, etc., are well explained on the bases of this third type of grouping.

We therefore consider these three types of groupings as descriptions of different outlooks on a given conceptual context in terms of mental structural images resulting from appropriate cognitive activity at the first three levels of thinking according to van Hiele.

CONCLUSION

The study of groupings in mathematics education has developed the originally obscure notion into a precise and powerful model for the description of products of cognitive activity in terms of mental structural images or cognitive maps of a given conceptual context. Different types of groupings illustrate and explain differences in reasoning and behavior at the first three levels of thinking as described by van Hiele and specify the notion of underlying structure, on which such differences are based. These different groupings also justify the rather lengthy process of transition from one level of thinking to the next through the gradual emergence of a new, more global outlook on the given context.

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A SYNTHESIS OF THE SOLO TAXONOMY WITH SKEMP'S MODEL OF MATHEMATICAL UNDERSTANDING WITHIN THE FRAMEWORK OF SKEMP'S MODEL OF INTELLIGENCE

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RÉSUMÉ

La taxonomie SOLO (Structure of Observed Learning Outcomes) est le résultat de plusieurs années d'études de John Biggs et Kevin Collis en Australie. Les cinq niveaux de complexité structurale qui émergent de leurs travaux sont dits préstructural, unisturctural, multi-structural, relationnel et abstrait élargi (extended abstract).

Quoique cette taxonomie soit d'abord conçue comme un outil d'évaluation de la qualité de l'apprentissage, l'intégration qui a été faite des niveaux de SOLO à la théorie développementale de Piaget a conduit à un modèle de développement cognitif semblable, sur plusieurs aspects, au modèle de compréhension mathématique de Skemp (en particulier) et à son nouveau modèle d'intelligence (en général). Skemp définit quatre types de compréhension mathématique: instrumental, relationnel, logique et symbolique. Il associe deux modes d'activité mentale à chacun de ces types: les modes intuitif et réflexif. Son hypothèse générale est que l'apprentissage intelligent peut s'expliquer par le biais d'un système directeur mental de second ordre ayant comme fin l'amélioration des schémas et plans d'action élaboré par le système directeur de premier ordre.

La taxonomie SOLO fournit à l'éducateur un cadre pratique pour classer les productions des étudiants sur une base qualitative (par opposition à une mesure quantitative). Les modèles de Skemp offrent à l'éducateur une façon d'expliquer comment et pourquoi l'étudiant en est venu à engendrer une production de telle qualité particulière et propose des stratégies d'enseignement pour améliorer la qualité de la production, c'est-à-dire, amener l'étudiant à un niveau plus élevé de SOLO.

SKEMP'S MODEL OF INTELLIGENCE - A GENERAL FRAMEWORK

In his opening address to the Third Annual Meeting of PME/NA, Skemp (1981) emphasized the need for both micro and macro models in the current research on mathematics learning. Skemp's own Model of Intelligence (1979a) provides the macro-framework for this synthesis.

Underlying Skemp's Model of Intelligence is the assumption that much of human behavior is goal directed. Figure 1 illustrates the two levels of director systems hypothesized by Skemp.

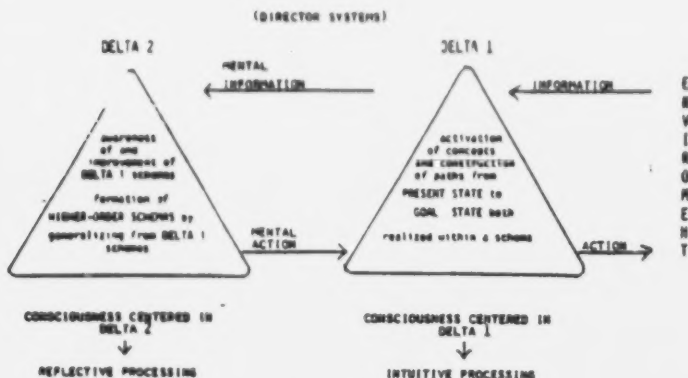


Figure 1: Skemp's Model of Intelligence

Intelligent learning includes the construction, testing and improvement (by Delta-two within Delta-one) of the schemas and plans which Delta-one must have to do its job. When consciousness is centered in Delta-two these activities are carried out by a process of reflection on one's schemas. When consciousness is centered in Delta-one Delta-two activity takes place unconsciously and the process is termed "intuitive".

The quality of schema available directly affects the quality of learning which can take place. Schemas can be regarded as networks of connected concepts. The concepts may be linked either associatively (A-linked) or conceptually (C-linked). The sequence of

letters in the alphabet is an example of an A-linked schema (the sequence must be learnt by rote). The sequence of numbers: 2-5-8-11-14 have a C-link — a difference of three.

SKEMP'S MODEL OF MATHEMATICAL UNDERSTANDING

Based on his model of intelligence, Skemp(1981) presented a synthesis of models of understanding which had been developed by several researchers on both sides of the Atlantic. Figure 2 illustrates the emergent model of mathematical understanding.

	TYPES OF UNDERSTANDING			
MODES OF MENTAL ACTIVITY	Instrumental	Relational	Logical	Symbolic
Intuitive	I1	R1	L1	S1
Reflective	I2	R2	L2	S2

Figure 2: Skemp's Model of Mathematical Understanding

According to Skemp, "To understand something means to assimilate it into an appropriate schema" (1971, p.46). The different kinds of understanding are thus generated by assimilation into different kinds of schema:

Instrumental understanding is the product of A-linked schemas produced by rote memorization resulting in fixed plans of action.

Relational understanding is the product of C-linked schemas.

Logical understanding is described by Skemp (1981) as "the ability to demonstrate that what follows is of necessity by a process of logical inference (formal proof)." This requires the existence of a higher-order schema, incorporating a system of logic with the relationally linked mathematical concepts.

Symbolic understanding is defined as "a mutual assimilation of a mathematical structure and an appropriate symbolic structure, dominated by the mathematical structure" (Skemp, 1981). The two modes of mental activity in Figure 2 refer to the intuitive and reflective processing described in the previous section.

THE SOLO TAXONOMY

Biggs and Collis (1982), while analyzing the responses of hundreds of elementary, high school, and college students in several subject areas, found that the assumptions of developmental stage theory did not hold. However, what did emerge from their analyses was a progression in the structural complexity of students' responses; the different levels of complexity form the basis of the SOLO taxonomy. Table 1 lists each SOLO level together with the amount and type of information required by the student in order to produce that particular level of response. The "relating operation" given in column three refers to the different ways in which the student relates to the given information.

TABLE 1
LEVELS OF THE SOLO TAXONOMY

LEVEL	INFORMATION REQUIRED	RELATING OPERATION
Prestructural	None available - cue and response confused	Denial, tautology
Unistructural	Only <u>one</u> relevant datum	Can "generalize" only in terms of one aspect
Multistructural	Several isolated relevant data	Can "generalize" only in terms of a few limited and independent aspects
Relational	Relevant data and interrelations	<u>Induction</u> . Can generalize within <u>experienced</u> contexts
Extended Abstract	Relevant data, interrelations and hypotheses	<u>Deduction and Induction</u> . Can generalize to situations not experienced

Note: Adapted from Biggs and Collis, 1982, pp.24-25

Biggs and Collis (1982) hypothesized the repetition of the SOLO levels within each major Piagetian stage of development, which gave rise to a new model of Learning Cycles. They extrapolated within this model to include higher stages of development termed "second-order formal," "Third-order formal" etc. The new model is illustrated in Table 2.

TABLE 2
A MODEL RELATING LEARNING CYCLES AND DEVELOPMENTAL STAGE

MODES OF FUNCTIONING	LEVELS OF LEARNING		CONSERVATION OF:
SENSORIMOTOR	UNISTRUCTURAL	MULTISTRUCTURAL	PRESTRUCTURAL
	RELATIONAL		
INTUITIVE	EXTENDED ABSTRACT → UNI		OBJECTS
	PRE	MULTI REL	
CONCRETE	UNI MULTI REL	← EXT ABS	CLASSES
		PRE	
FORMAL (FIRST ORDER)	EXT ABS	→ UNI MULTI REL	SYSTEMS
	PRE		
FORMAL (SECOND ORDER)	UNI MULTI REL	← EXT ABS	THEORIES
		PRE	
ETC.	EXT ABS	→ UNI	META THEORIES

LEARNING CYCLES:

PRESTRUCTURAL - NO DATA AVAILABLE

UNISTRUCTURAL - FORMATION OF A DATUM

MULTISTRUCTURAL - FORMATION OF PARALLEL DATA

RELATIONAL - INTEGRATION OF DATA

MODALITY SHIFT - EXTENDED ABSTRACT -

FORMATION OF HIGHER ORDER CONCEPT

Note. Adapted from Biggs and Collis, 1982, p.216

In Table 2 the developmental sequence is referred to as "modes of functioning" (column 1). The second column represents the learning cycle from simple to complex (unistructural to relational) within each mode. The five SOLO levels are implicit to each mode (as

indicated by the dotted boundaries). An element in an earlier mode is "prestructural" with regard to the present mode of functioning. Achieving an "extended abstract" response indicates a modality shift to the next mode. This response would now be regarded as "unistructural" within the higher mode of functioning. The last column gives an example of the type of conservation achieved at each modality shift.

THE SYNTHESIS

In relating Skemp's Model of Intelligence to the SOLO model (Table 2), an explanation of how a student moves from one level in the learning cycle to the next can be generated: The form of the student's Delta-one schemas has to change. The SOLO levels, in fact, can be regarded as indicating different kinds of schema which have qualitatively different types of connections as illustrated in Table 3.

Change in the form of a person's Delta-one schemas can only be brought about by the action of the person's own Delta-two. To move from prestructural to unistructural requires the formation of a concept of the appropriate order. The movement from unistructural to multistructural involves awareness of several isolated concepts. The transition from multistructural to relational involves the formation of C-links between the concepts such as the order relations "more than" and "less than" on the counting numbers. In order to generate an extended abstract response the individual needs available several C-linked schemas formed over time by experiencing similar problems in different contexts (multi-embodiments). The individual must be aware of these several schemas and by reflecting on them abstract the general principle which links them together. In this way a higher-order concept is formed which is then brought into action by Delta-one and applied to the problem. The individual's response is thus indicative of a higher mode of functioning, and the higher-order concept is the isolated concept schema necessary for a unistructural response within the new mode of functioning.

TABLE 3

	<u>SOLO LEVELS OF RESPONSES</u>	<u>SKEMP'S QUALITIES OF SCHEMAS</u>
BRUTIVE	PRESTRUCTURAL - DOES NOT ENGAGE THE TASK OR GIVES COMPLETELY UNASSOCIATED DATA	NO APPROPRIATE SCHEMA AT THE (CONCRETE) STAGE
	UNISTRUCTURAL - FORMATION OF A SINGLE DATUM	ISOLATED CONCEPTS
CONCRETE	MULTISTRUCTURAL - ACQUISITION OF PARALLEL DATA WHICH ARE NOT RELATED	ASSOCIATIVE LINKS (A - LINKS)
	RELATIONAL - INTEGRATION OF DATA	CONCEPTUAL LINKS (C - LINKS)
FORMAL	EXTENDED ABSTRACT - SUCH HIGHER - ORDER CONCEPTS ARE THE DATA FOR THE NEXT CYCLE	HIGHER - ORDER CONCEPTS AND SCHEMAS

SOLO LEARNING CYCLES AND TYPES OF MATHEMATICAL UNDERSTANDING

Unistructural and multistructural responses are indicative of instrumental understanding when they are the forced product of rote memorization. Otherwise they are necessary steps in a process of conceptualization which may eventually lead, through relational understanding, to a higher-order concept. Trying to force a jump from unistructural or multistructural to extended abstract produces a false modality shift which results in an instrumental learning cycle which is then repeated within the higher modality. A true modality shift comes as a result of a relational learning cycle (produced by involvement with and action on the subject matter) and carries with it the potential for a relational learning cycle in the higher mode.

of functioning. Reflective processing is necessary to achieve the modality shift.

Logical understanding indicates a modality shift to the "first-order formal" mode of functioning. However, the integration of mathematical concepts within a logical proof indicates a relational SOLO level of response in this higher mode.

Symbolic understanding is indicative of another modality shift as it requires the mutual assimilation of two system-level structures. The student would now be capable of dealing with theoretical mathematics (pure mathematics), although his/her response would initially be unistructural within the "second-order formal" mode of functioning.

The above correspondences add interiority to both models. The notion of instrumental understanding helps distinguish between different types of unistructural and multistructural responses. The SOLO model of learning cycles indicates the existence of several steps in the progression from relational to logical understanding. The idea of symbolic understanding supports the extrapolation within the SOLO model to higher-order formal modes of functioning.

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ON CHILDREN'S QUANTITATIVE CONCEPT OF RATIONAL NUMBER:
CONSTRUCT AND ESTIMATE THE SUM

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RESUME

Ce texte est un des nombreux rapports issus du "Rational Number Project" touchant la question de l'élaboration d'un concept quantitatif de nombre rationnel chez les enfants de quatrième et cinquième année. Comme mesure possible de ce concept, cette étude rend compte des performances des enfants dans une tâche d'estimation pour laquelle le temps était limité et dans laquelle ils avaient à construire, à partir de chiffres choisis, une somme de deux nombres rationnels qui devait être aussi près que possible de l'unité. Deux versions de cette tâche furent employées. A partir de l'écart-moyen à l'unité dans trois essais de la tâche, les enfants furent classés comme exécutants forts, moyens ou faibles. Les explications obtenues dans des entrevues individuelles, permirent de caractériser les mécanismes cognitifs exhibés par les exécutants forts comme étant le recours à l'application souple et spontanée des concepts d'ordre des nombres rationnels et d'équivalence de fraction et le recours à un procédé identifiable d'estimation (utilisation d'un point de référence). Les exécutants faibles furent caractérisés par l'absence manifeste de ces mécanismes cognitifs ou par des manifestations limitées, incorrectes ou grossières de tels mécanismes cognitifs.

A good understanding of fraction size seems important for children to use rational-number concepts in computation and problem solving; however, assessment of this is difficult. This paper considers an estimation task of constructing a two-addend rational-number-sum close to 1 as a measure of the quantitative concept of rational number, and is an elaboration of results presented by Hachsmuth, *et al.* (1983).

The research literature concerning children's ability to estimate arithmetic computations is sparse and is especially so for rational-number computation. A National Assessment of Educational Progress (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980) found that only 24 and 37%, respectively, of 13- and 17-year-olds selected correctly from among 1, 2, 19, and 21 to estimate $12/13 + 7/8$; most frequent responses were 19 and 21. Trafton (1978) identified four estimation processes and indicated that "use of a reference point" promotes reflective thinking. Streefland (1982) investigated the effect of estimation on ratio and proportion tasks; results suggest that such reflective thinking, which can be provoked and developed by the demands for estimating, organizes the problem domain and prestructures the solving process. Payne and Seber (1959) and Buchanan (1978) indicate that the ability to estimate requires early experience with concepts of more, less, between, and equals. Bright (1976) suggests the ability to estimate measurements requires the development of points of reference. Estimation is indispensable to the notion of "quantitative literacy" (Siegel & Goldsmith, 1982). A positive correlation of 0.74 between college students' scores on the School and College Ability Test (SCAT) quantitative ability subtest and an author-constructed test of estimation ability was reported by Levine (1982).

The research literature suggests that children's ability to make good estimates of arithmetical computations is related to their concept of number size. This paper considers a competitive rational-number estimation task of constructing, under time constraints and from a limited set of numerals, a two-addend rational-number sum close to 1. The purposes of this investigation were to (a) gain information about children's accuracy on such a task and (b) gain insights into the

cognitive mechanisms exhibited by successful children.

THE STUDY

The present study was conducted by the Rational Number Project during 1982-83. Instruction in a 30-week teaching experiment was based on the multiple-embodiment principle (Dienes, 1971) and included the use of several manipulative materials, representational modes, and rational-number constructs (Behr, *et al.*, 1980). An important aspect was that translations between different modes of representations which subjects frequently were to make during instruction would facilitate the abstraction of rational-number ideas (Lesh, *et al.*, 1980). The study presented in this paper is part of a larger set of studies aimed at assessing children's quantitative notion of (positive) rational number (see Wachsmuth *et al.*, 1983). Subjects were eight children at DeKalb, selected to reflect the full range of ability, and a classroom size group of 34 homogeneously-grouped middle-ability children at Minneapolis-St. Paul. Both groups were instructed and observed during their fourth and fifth grades. Data from two clinical interviews with the DeKalb children and with eight from Minneapolis form the basis of this study.

Version 1 of the Estimate-the-Sum task consisted of six cards with the numerals 1, 3, 4, 5, 6, 7 and a form board as shown in Figure 1. Version 2 used cards with 11, 3,

4, 5, 6, 7. Version 1 was presented as part of a video-taped interview following 20 weeks of instruction; both were presented



Figure 1

after 27 weeks of instruction. Subjects were directed to "put number cards inside the boxes to make fractions so that when you add them the answer is as close to one as possible, but not equal to one." To discourage the use of algorithms, subjects were encouraged to estimate, and a time limit of one minute was imposed on the task.

RESULTS

The percentage deviations of constructed responses from 1 varied from 2.38% to 491.71% with an average percentage deviation over all subjects on all tasks of 52.75%. Based on children's explanations, the

responses were classified into four categories plus an "other" category. Sample responses exhibiting subjects' thinking on the tasks are given with descriptions of the categories.

CATEGORY ER (Estimate by correct comparison to a Reference point). Response explanations indicate a successful attempt to estimate the constructed rational-number sum by using one-half, 1, or some other self-chosen fraction as point of reference. The spontaneous use of fraction equivalence and rational-number order is evident.

BERT: [Using 1 3 4 5 6 7 constructs $5/6 + 1/7$]...well uh,...five ...five-sixths [pointing to 5/6] is one piece away from the unit, and a seventh is just a little bit smaller, so that could fit there (i.e., between 5/6 and 1).

KRISTY: [From 11 3 4 5 6 7 constructs $6/11 + 3/7$ and changes to $5/11 + 3/7$] ... Well five and a half is half of eleven [pointing to 5/11] and [pointing to 3/7] three and a half is half of seven, so it would be one away from ... (and I changed 6/11 to 5/11) ... because [pointing to 6/11] that would be a little more, that's [pointing to 3/7] less than one (-half) ... I was afraid they'd get exactly one.

BERT: [From 11 3 4 5 6 7 makes $3/6 + 5/11$] ... Three-sixths is half a unit and ... if it was five and a-half-elevenths [pointing to 5/11] that would be half; and a half (i.e., one-half-elevenths) would be very thin.

CATEGORY MC (Mental algorithmic Computation). Response explanations indicate that the subject did mental computation to carry out a correct standard algorithm (e.g. common denominator) to determine the actual sum of the generated fractions. The spontaneous use of fraction equivalence and rational-number order is evident.

KRISTY: [Using 1 3 4 5 6 7 makes $\square/3 + \square/4$, then changes to $1/3 + 4/5$] ... If you find the common denominator, twelve; but ..., and then four times one would be four [explaining the change $\square/4$ to $4/5$] , but then three times ... I didn't have a two or anything (among the number cards given and remaining) and I used up my three so

CATEGORY ERI (Estimate by Incorrect, gross, or uncertain comparison to a Reference point). Response explanations indicate that the subject attempted to estimate the constructed rational-number sum by using one-

half, 1, or some other self-chosen fraction as a point of reference. Little or constrained understanding of fraction equivalence and rational-number order is evident.

JESSIE: [From 1 3 4 5 6 7 makes $1/3 + 5/6$] ... [points to 5/6] that's less than a half and ... if you add wait ... if you add one-third it would be bigger ... [points to 5/6] that ... that's bigger than one-half ... that's (i.e. $1/3$) less than one-half ... (and $5/6$) is less than half, wait bigger than, less than half ...

MACK: [From 1 3 4 5 6 7 makes $5/6 + 3/4$] I just thought about equivalent fractions ... like ... wait ... like you take closest to one you can get, three-fourths 'cause it's only one (i.e. one-fourth) away (from 1), and the same with this one [pointing to 5/6].

CATEGORY MCI (Mental algorithmic Computation based on Incorrect algorithm). Responses indicate that the subject used mental computation based on an incorrect algorithm to compute the actual sum.

TED: [From 1 3 4 5 6 7 makes $5/6 + 4/7$] ... Well first I thought, I tried to figure out what would come closest to one and I found out that five-sixths and four-sevenths would come the closest ... 'cause I used the top number ... (If I added them) nine-thirteenths.

JESSIE: [From 1 3 4 5 6 7 makes $3/7 + 4/6$] ... Three plus four is seven and that's [pointing to 7 and 6] thirteen ; it would be one-thirteenth close to one.

CATEGORY G (Gross estimate) Response explanations suggest that the subject made a gross estimate of each rational-number addend, but did not make a comparison to a reference point, and did not use fraction equivalence or rational-number ordering.

TED: [From 1 3 4 5 6 7 makes $3/11 + 4/7$] ... I wanted to use up the little pieces for the top ... then use the highest number of pieces for the bottom ... well, if I ever thought if it was equal, or one's less or greater and stuff, I always have to be greater than the top number.

Responses in Categories ER and MC represent the highest performances as measured by the deviation of the constructed sum from 1. Responses categorized in Category ER ($n = 11$) had an average deviation of

6.16%, those in Category MC ($n = 3$) 17.79%, and the percentage across Categories ER and MC was 8.65%. For Category ERI ($n = 4$), the average deviation was 26.96% and across Categories ER, MC and ERI ($n = 18$) it was 12.72%. This is contrasted to the average deviation in categories other than ER, MC, and ERI ($n = 18$), which was 97.65%. (Because of incompleteness, data of two subjects were omitted in this calculation).

DISCUSSION

The subjects had received some instruction on estimation of whole-number sums by rounding, but no formal instruction on estimating the sum of two rational-numbers. They had received extensive experience with rational-number order and fraction equivalence. In some lesson near the assessments relevant for the present study, class activity included one in which children were given a fraction, and they in turn were to give fractions successively closer to the one given.

Among children whose performance on the Estimate-the-Sum task was rated as high, 11 of 14 responses given by five of the 14 subjects considered were in Category ER and the remaining three responses in MC (no response was given by one child in one presentation of the task). That is, children whose performance on this task was rated as high, almost uniformly used estimation; moreover, the estimation process was that of using a "reference point" (Trafton, 1978). They also displayed ability for spontaneous and flexible application of fraction order and equivalence concepts. It appears that a combination of skills in estimation and order and equivalence is important for a good quantitative concept of rational number.

Children whose performance was rated as low on the Estimate-the-Sum task gave no responses in the ER or MC categories but all among MCI, ERI, G, and "other". These responses show an absence of application of order and equivalence concepts and of use of accurate estimation processes such as use of a reference point.

OTHER OBSERVATIONS

While care must be exercised about broad generalizations given the small sample size, it is useful to make additional observations about other cognitive characteristics of children who exhibited Category ER and MC responses as compared to children whose responses fall in other

categories. Bert's explanations, in general, exhibit considerable imaginal memory. Exemplified by his responses in Category ER above, there is evidence to suggest that he imagines episodic experiences associated with the manipulative-based instruction. Remarkable is Bert's ability to associate oral symbols for mathematical entities with the imaginal units ("five-sixths is one piece away ..."), exhibiting his ability to translate between ideas expressed via manipulatives and ideas expressed in mathematics symbolism. Kristy's ability to store a sequence of memory units together with tremendous mental symbol-manipulation capability is evident in her Category MC response. She has ability to "preview" an entire algorithm sequence, with her memory needing to deal only with numerical entries of the algorithm; the process is automatic. Kristy and Bert's explanations further demonstrated excellent applications of order and equivalence concepts while some other students displayed an inability to apply these concepts in the estimation task.

In summary, responses which placed subjects in the high performance level on the Estimate-the-Sum task are characterized by a spontaneous and flexible application of fraction order and equivalence concepts and (in most cases) accurate application of an identifiable process of estimation (use of a reference point). Responses which placed subjects in the low performance level are characterized by indicating constrained use of fraction order and equivalence concepts and very uncertain or inaccurate, if any, use of an estimation process. Thus, a high level of understanding, and applicability, of fraction order and equivalence appears necessary for an ability to give estimates of rational numbers and rational-number sums.

As was mentioned, Streefland (1982, p. 197) suggested that estimation can "elicit a global orientation on the problem set, which organises the problem domain and prestructures the solving procedure." We see this prestructuring behavior in Bert and Kristy (see Category ER). This ability to see a global organization of a problem domain was observed (Behr, Wachsmuth, Post, and Lesh, 1983), in children who were able to use manipulative aids flexibly, translate between different aids, and display thinking which had progressed from manipulative

dependent to manipulative independent. These issues -- estimation, quantitative concept of number, flexibility of thought, and translation between representational modes -- appear to be interrelated and demand the attention of researchers.

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TEACHING SIMILARITY CONCEPTS IN THE MIDDLE GRADES

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RESUME

Le NSF-DISE fournit les fonds au Middle Grades Mathematics Project pour produire les unités d'un programme d'études pour les classes de cinquième à huitième (les élèves de onze ans à quatorze ans). Les unités représentent une approche unique dans l'enseignement des mathématiques pour ces classes.

Le Similarity Unit initie les élèves aux idées ayant rapport aux similarités par une succession d'activités qui commencent avec un élastique tendu. Ensuite, on présente les systèmes coordonnés, les familles des rectangles et des triangles semblables, les formes qui se répètent (repeating tiles), les projections des pointes et les applications. Dans chaque cas on fait ressortir le caractère de l'augmentation par l'étude de l'agrandissement des dimensions et l'effet produit sur la mesure linéaire et la superficie.

Le Similarity Unit a été évalué dans les écoles urbaines, de banlieue et rurales. Il y avait 24 enseignants et 56 classes d'élèves. L'analyse du progrès des élèves a indiqué qu'il y a eu une amélioration considérable de la performance par classe mais non par sexe.

THE MIDDLE GRADES MATHEMATICS PROJECT

The MGMP is a curriculum development project funded by NSF - DISE, to develop units of high quality mathematics instruction for grades 5 through 8.

Each unit:

- is based on a related collection of important mathematical ideas,

- provides a carefully sequenced set of activities which lead to an understanding of the mathematical challenges,
- helps the teacher foster a problem solving atmosphere in the classroom,
- uses concrete manipulatives, where appropriate to help provide the transition from concrete to abstract thinking,
- utilizes an instructional model which consists of three phases...launching, exploring, and summarizing,
- provides a carefully developed instructional guide for the teacher,
- requires two to three weeks of instructional time.

The Similarity Unit is the fourth unit designed, classroom tested and evaluated by the MGMP—The other three topics being Spatial Visualization, Factor and Multiples (Ben-Haim, et al, 1982), and Probability (see this issue).

SIMILARITY

Twenty years ago, P.M. and D. Van Hiele, theorized that students pass through a sequence of developmental levels in their acquisition of the content and logical framework of geometry.* They concluded that students need to be given many concrete geometric experiences before they are prepared to study geometry in a formal way. Several very recent research studies in this country have confirmed the existence of these cognitive levels and the need for a long term exposure to geometry in an experimental, informal way before the traditional high school formal geometry.

From the many important topic areas in geometry, the MGMP has selected similarity and its close connections with equivalent fractions as the focus of a unit. The cognitive and the educational aspects of similarity concepts will be analyzed as reasons for this selection.

The cognitive aspect. The concept of similarity seems to play an important role in the cognitive development of the child. Piaget

* A summary of the Van Hiele levels can be found in an article by Burger, et al. (1981).

(1967) describes five experiments related to the concept of similarity, and reports three developmental phases that follow roughly the evolution of the "middle geometries" in Klein's Erlanger Program (Fig.1).

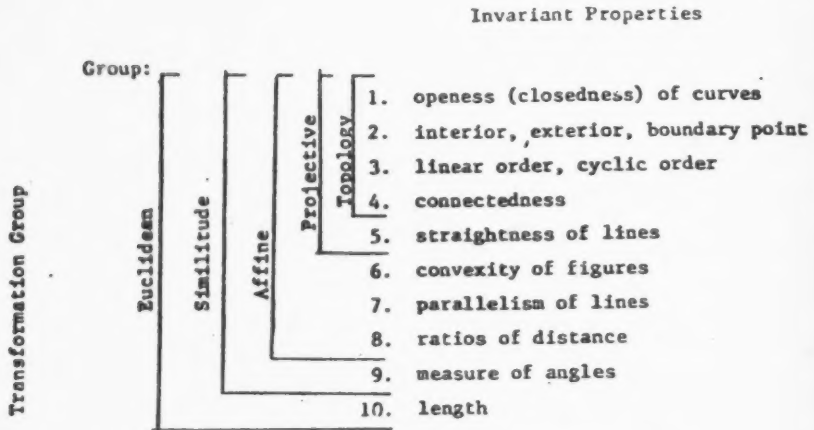


Fig.1 Properties invariant under transformation groups.
(adapted from Martin, 1976, p. 95)

The use of proportional reasoning in most similarity tasks allows for further cognitive implications. Proportional reasoning is considered by Inhelder and Piaget (1958) as one of the eight schemes in the last stage of an individual's development of thought—the formal thought stage. Moreover, Martorano (1974) found that of the six schemes tested, the proportionality scheme was more difficult than all others, except mechanical equilibrium. Linn and Swiney (1978) indicates a strong relationship between proportional reasoning and general ability.

Educational Implications. Relating specifically to studies on instruction in similarity, Fuson (1978) draws the following conclusions:

Similarity ideas are included in many parts of the school curriculum. Some models for rational number concepts are based on similarity; thus, part of student's difficulty with rationals may stem from problems with similarity ideas. Ratio and proportion are part of the school curriculum from at least the seventh grade on, and they present many difficulties to the student. Standardized tests include many proportion work problems. Verbal analogies ($a:b::c:d$) form major parts of many intelligence tests. Similar geometric shapes would seem to provide a helpful mental image for other types of proportion analogy situations. Training studies of teaching experiments concerning ways to teach geometric similarities and ways to generalize the solution of geometric proportions to other types of proportion would be valuable. (p. 259)

However, as research shows, performance on proportional reasoning tasks in general, and on similarity in particular are far from satisfactory. Relating to the inadequacy of instruction at the junior high school level, Wollman, and Karplus (1974) point out that "ratios are introduced as fractions and proportions as equivalent fractions, to be handled by certain standard procedures of multiplications and divisions" (p.610).

We have chosen similarity because we consider it to be one of the more basic ideas in understanding the geometry of indirect measurement, scale drawing, scale models, and the nature of growth. Similarity provides an excellent opportunity to build more geometric meaning into the idea of equivalent fractions. When one constructs a scale model or drawing of a real object the ratios of corresponding segments or distances for the scale model and the real object can be expressed as equivalent fractions.

The relationship between the lengths in similar figures and the areas

of these similar figures is explored in the unit through looking at concrete examples and the patterns that come out of these examples. Looking at patterns and making generalizations from patterns is an important and appropriate problem solving heuristic for students at this age. The culmination of the Similarity Unit is an activity which poses for students a real application problem, the challenge of measuring the heights of inaccessible objects by using the mathematical concepts and models of the unit. All of these ideas are presented through concrete hands-on explorations for the students.

Many materials and activities developed in the past two decades have failed to be successfully implemented because the spirit of the materials is lost when they are put into written textbook form. MGMP has put its activities and ideas in student activity pages and a very detailed teacher guide to help a teacher understand and manage an activity based geometry unit. We believe that this will greatly increase the success of teachers and students with our materials.

PURPOSE

The investigators assert that tasks involving similar shapes satisfy this condition, and more generally, that the MGMP satisfies the criteria for curriculum design and evaluation emphasized in the above quoted recommendations.

The purpose of this study is to detect the extent to which the MGMP Similarity Unit will help overcome the cognitive difficulties which undoubtedly do exist. More specifically, patterns in gains in achievement in similarity tasks will be considered, especially in regard to grade and sex differences.

METHODOLOGY

The study was conducted during winter, and spring, 1983. Pre and post instruction data was collected using a 25 item Similarity Test.

The subjects of the study were 1187 6th through 8th graders from Lansing, Michigan area schools. The participating schools were categorized according to site into (1) urban (inner city), having mainly minority or low SES students, (2) suburban, and (3) rural, the two with a mainly white, middle class population of students.

The 24 math teachers who taught the unit were provided with the instructional material, a teacher guide, and a two-hour workshop. The instruction was, on an average, three weeks long. The Similarity Test was administered during regular school days by the classroom teachers. The statistical analyses included analysis of variance and repeated measure.

RESULTS

Diagram 1 presents the data on similarity achievement organized by site, by grade, and by sex. Findings which are significant at a .05 confidence level will be reported.*

1. At all sites, and for all grades a significant achievement gain occurred. Due to the very short period of instruction these gains cannot be attributed to confounding effects such as maturation.
2. In all cases girls scored lower than boys, both before and after instruction. Significant sex differences in achievement were observed at the suburban and rural sites, but boys and girls showed no significant differences in gains.
3. In order to detect grade differences, planned comparisons of grades 6 vs. 7, and (6,7) vs. 8, were conducted. In both suburban, and rural sites, no significant differences in gain were detected between grades 6 and 7. However, grades 6 and 7 combined together gained significantly more than the 8th graders.

* A drastically shortened period of instruction at the urban site caused difficulties in the interpretation of the corresponding data. Therefore, these results will not be analyzed.

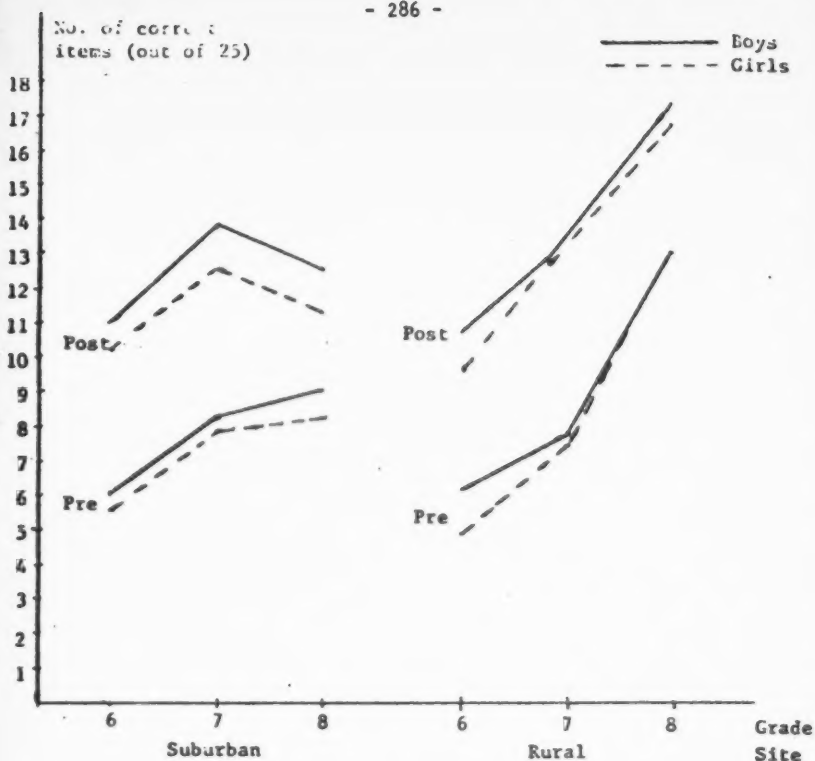


Diagram 1. Achievement on the Similarity Test by site, by grade, by sex.

CONCLUSIONS

The results of this study suggest, that (1) achievement in similarity tasks can be improved through instruction, (2) due to the relatively weak achievements (the posttest average scores ranged between 38% and 70%), a spiral approach to the topic may be needed, (3) boys and girls seem to profit equally from the treatment (with girls scoring slightly less than boys), and (4) 6th and 7th graders might profit more from the instruction of the unit, as compared to the 8th graders. A possible explanation of (4) is the better attitudes of younger children to an approach considered at the higher grades to be "playful and childish."

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RESULTS OF A TWO-YEAR STUDY ABOUT THE EFFECTS OF CALCULATORS
ON ACHIEVEMENT IN 5th AND 6th GRADES

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Résumé

De septembre 1980 à juin 1982, les auteurs ont mené une étude longitudinale dans 20 classes d'élèves provenant de cinq commissions scolaires de la Région du Grand Québec. L'étude visait deux objectifs: (1) vérifier si l'utilisation de calculatrices a ou n'a pas d'effet nocif sur le rendement des élèves en mathématiques à la fin du primaire; (2) amorcer une recherche à propos des effets potentiels des calculatrices sur l'habileté des élèves à faire des estimations de résultats numériques, leur habileté à découvrir des régularités numériques, leur habileté à résoudre certains problèmes à données manquantes ou superflues, leur habileté à procéder par essais et erreurs pour résoudre certains problèmes numériques.

Les auteurs donnent ici les résultats qu'ils ont trouvés en rapport avec le premier objectif. Ils justifient également pourquoi ils ont senti le besoin d'entreprendre cette étude alors que déjà un grand nombre de travaux semblables avaient été réalisés antérieurement. En bref, il ressort de l'étude que l'emploi de calculatrices ne semble pas nuire au rendement en calcul des élèves de la fin du primaire. (Recherche subventionnée par le Fonds FCAC, par la D.G.D.P. du ministère de l'Éducation et par l'U. Laval)

During 1979, the authors decided to conduct a long-term study⁽¹⁾ with the following two main objectives:

- (1) to ascertain whether or not the use of calculators would have detrimental effects on students' mathematical achievement at the end of the elementary school;
- (2) to start investigations about effects calculators might have on the following students' abilities: to make estimations of numerical results; to discover numerical patterns; to solve arithmetical word problems with missing or with extraneous data; to use a trial-and-error strategy to solve certain numerical problems.

The study was actually conducted in the Greater Quebec City area, with 20 classes of pupils having entered 5th grade in September 1980 and completed 6th grade in June 1982, half of the classes having been initially assigned at random to the calculator treatment and the other half to the non-calculator treatment. In this paper, only the results concerning the first objective above will be reported.

At the time the study was planned, the authors were aware of the existence of a great number of previous studies on the basis of which M. Suydam wrote: "The calculator does not appear to affect achievement adversely. In all but a few instances, achievement scores are as high or higher when calculators are used for mathematics instruction (but not on tests) than when they are not used for instruction."⁽²⁾ But they were also aware of many criticisms that have been made concerning a majority of existing studies about effects of

(1) This research was supported by the Québec "Fonds FCAC pour l'aide et le soutien à la recherche", the Québec Ministry of Education (D.G.D.P.) and Laval University.

(2) Cf. Suydam, Marilyn M. The Use of Calculators in Pre-College Education: A State-of-the-Art Review. Columbus, Ohio: Calculator Information Center, May 1979.

calculators. For example, Wheatley and Shumway⁽³⁾ observed that most of the studies suffer from serious design, procedural, or interpretation problems such that few valid conclusions can be made. M. Suydam⁽⁴⁾ went beyond this seeing a greater concern than these flaws: most studies touch on broad, general questions of achievement relative to untold or vague use of the calculator. Parkhurst⁽⁵⁾ concluded that "Many of the studies reported are disappointingly similar in the fact that they are not concisely reported, are ambiguous, and have serious methodological flaws. Therefore, the studies lack credibility causing the validity of the conclusions to be questioned." On the other hand, in his review of thirty-four empirical studies, Roberts⁽⁶⁾ observed that "Interpretation of the results of many of the studies was hampered by defective research designs including assignment of students to conditions, contamination of treatment with control groups, control of the teacher variable, and the lack of use of calculators on the posttests." Another recurring criticism concerns the treatment length: most studies done to date have been too short in length (e.g. a few hours or a few weeks) to warrant valid conclusions.

Accordingly, the authors felt it would be worthwhile to conduct one more study about the effects of calculators on students' achievement, under the condition, however, that it would be a long-term study, that special efforts would be made to prevent major flaws reported about previous similar studies and that the objectives would be broad-

(3) Cf. Wheatley, G.H. & Shumway, R.J. Impact of Calculators in Elementary School Mathematics. Final Report. (ERIC Reproduction Service, No. ED 175720) July 1979

(4) Cf. Suydam, Marilyn N. Investigations with Calculators: Abstracts and Critical Analysis of Research. Columbus, Ohio: Calculator Information Center, Jan. 1979

(5) Cf. Parkhurst, S. Hand-Held Calculators in the Classroom: A Review of the Research. (ERIC Reprod. Service, No. ED 200416) 1979

(6) Cf. Roberts, D.M. The Impact of Electronic Calculators on Educational Performance. Review of Educational Research, Spring 1980, Vol. 50, No. 1.

ened in order to include the second objective already mentioned.

Five schoolboards were chosen in the Greater Quebec City area in order to get a representative sample. A call was made for teachers who would teach in 5th grade during 1980-1981 and who would volunteer to participate in the study. Their number was reduced to twenty, after a few ones were randomly eliminated in schools where there were too many volunteers. Out of the remaining 20 teachers, 10 were *randomly* assigned to the calculator treatment, the other 10 serving as a control, with one restriction however: all classes from a given school had to be assigned to the same treatment, in order to avoid contamination of treatment with control classes. Thus, choosing classes as units, each of the experimental group and the control group consisted of 10 classes taught by the selected teachers during 1980-1981. During 1981-1982, it was possible to keep most of the 20 classes intact when they were promoted to 6th grade. However new teachers were in charge of the classes. (Those teachers had accepted to participate *before* knowing whether they would get a calculator class or a non-calculator class.) Special information sessions for all participating teachers and for local collaborators (mathematics supervisors in the schoolboards concerned) were organized by the authors in October 1980 and in October 1981. In each of the classes participating, special care was taken to inform the parents of the pupils about the objectives of the study and about the precautions being taken.

Special efforts were made to conduct this study in such a way that the setting and the functioning of each class involved remain as "natural" as possible. Even in experimental (calculator) classes, there was a regular teacher (not an experimenter), teaching with his/her usual style, trying to cover the regular curriculum and using the regular textbook. However it was felt necessary to introduce a minor disturbing element in order to better achieve both the first and the second objectives of the study, that is to ask teachers to use specially prepared sets of word problems as a *complement to the regular textbook*. The main reason for doing that was that the textbooks used in the schools did not contain enough exercises that could *justify*

the use of a calculator in the experimental group. Of course, the teachers of both experimental and control classes had to use those sets of problems, following the same instructions: 1. Give at least two of those problems to your students every week. 2. Devote an average of one hour per two weeks to such problems. At regular intervals, all teachers were requested to fill special control forms about the way they had used the special sets of problems with their pupils. (Examples of such problems will be shown during the presentation.)

In each of the experimental classes, one calculator was available for each of the pupils. Model ROYAL L-816 (with memory lock and automatic power off) was used uniformly. Extra machines were available for teachers and for replacement. In October 1980 and in October 1981, special sessions were organized by the authors for teachers of the experimental classes and for local collaborators in order to discuss the functioning of the calculators as well as how and to what extent they could be used in the experimental classes in view of the objectives of the study. The general guideline agreed on was this one: *encourage the pupils to use their calculator (never force them!)* (a) *to solve problems from the special sets* (referred to above); (b) *to solve some problems of the regular textbook for which the use of a calculator makes sense and looks justifiable*; (c) *to check some results already obtained with paper and pencil or mentally*; (d) *occasionally to verify by means of particular examples some numerical rules or properties found in the regular program*. It was left to the teachers ^{to decide} at what time and how calculators would be made available to the pupils in the classroom. At regular intervals, teachers of the experimental classes were requested to fill special report forms concerning how, for what and for how long calculators had been used by the pupils in their class.

In order to compare the achievement of the calculator and of the non-calculator groups, three tests were administered: a pretest in October 1980, an intermediate test in June 1981 (end of 5th grade) and a posttest in June 1982 (end of 6th grade). Each time the same test

was used, that is a standardized test normally used by the Québec Ministry of Education when pupils leave the elementary school (6th grade). DURING EACH OF THOSE TESTS, PUPILS WERE NOT ALLOWED TO USE CALCULATORS (including pupils of the experimental group).

In order to make a statistical analysis of the test results, it was necessary to reduce the number of classes to 16: 8 experimental classes and 8 control classes. There were various reasons for that: some pupils had missed one or two of the tests, some others had moved to another school or class, some had not identified or filled their answer sheets properly at one test, one teacher got one questionnaire too late, etc., so that 4 classes had to be eliminated (including some that had become too small!). Then the authors had to randomly eliminate other students in order to get 16 classes of equal size ($16 \times 14 = 224$ pupils).

A 3-factorial single multivariate analysis of variance with a repeated measure on one factor was used to analyze the results of the 16 classes. The three factors considered were: the *test* factor (fixed and repeated three times), the *experimental condition* factor (fixed) and the *class* factor (random). It has not been necessary to test the homogeneity of the variance-covariance matrix with a Box's test. All significant results obtained with the F-test used remained significant when tested again with a conservative F-test. The 0.01 level was chosen as significance level.

The main result of the analysis is that the experimental condition factor did not have a significant effect. In other words, *whether calculators are used or not, there is no statistically significant difference in achievement on the test.*

On the other hand, the test factor had a significant effect. In other words, the achievement of pupils on the test increased over time — which, of course, is not a surprising result.

Moreover the class factor had a significant effect, which means that

the teacher obviously had a much greater influence on pupils' achievement than the experimental treatment.

A 3-factorial single multivariate repeated measures design on one factor was also used to analyze the results, but this time with the three following factors: the *test* factor (fixed and repeated), the *experimental condition* factor (fixed) and the *sex* factor (fixed). If a significance level of 0.01 is chosen, then the test factor had a significant effect, which is consistent with the preceding conclusions. A significant effect of the interaction (experimental condition) x (test) was also found. However this significant effect remains difficult to interpret, taking into account the fact that there is a significant class effect, a significant second-order class effect and even a significant triple interaction. It has therefore to be assumed that the significant interaction (experimental condition) x (test) may be due to the class factor.

This study seems to indicate that the use of calculators will not *harm* students' performance — at least at the 5th and 6th grade levels — when they are faced with doing calculations by paper-and-pencil methods. The authors are well conscious of, and recognize the fact already emphasized by Dennis M. Roberts (in reference 6) that this is a rather negative orientation: "Instead of examining potential positive impacts, the focus is on demonstrating the *lack* of negative effects. However, it seems more realistic to assume that calculators may have more positive benefits than negative. If that is the working assumption, then allowing one to use the calculator on the actual test seems the more useful research strategy." But, at the time the study was planned, they felt the need — as well as some pressure from the public — to *first* establish that calculators do not have detrimental effects on achievement, by means of a *long-term, well designed* study...

A detailed report will be available from the authors later this year.

THE UNDERSTANDING OF DIRECTED NUMBERS

by

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RESUME

Cet article traite de l'utilisation de modèles dans le but de favoriser la compréhension des entiers relatifs. Dans une première partie, nous abordons la question du rôle des modèles en général. Ensuite, nous présentons une classification des modèles des entiers relatifs. Nous y distinguons deux grandes catégories: les modèles fondés sur l'idée d'équilibre et ceux utilisant la droite numérique comme support. Nous y distinguons également les modèles symboliques et les modèles schématiques. Les premiers, comme leur nom l'indique, sont fondés essentiellement sur des symboles et leur qualité. Les seconds font appel aux images mentales inhérentes à la situation initiale caractérisant le modèle.

En conclusion, nous présentons quelques résultats d'une expérimentation réalisée récemment qui montrent que le modèle schématique s'est avéré supérieur pour les items d'addition. D'autre part, pour les items de soustraction, nous n'avons constaté aucune différence.

A) The use of models.

Models are often called *embodiments* or *iconic representations* which are used to convey more "concretely" some abstract mathematical idea. For DIENES and others, the principle of multiple *embodiments* insures the right construction of abstract notions and therefore guarantees good understanding. Recent research for our group (GEPREM¹) tends to show the opposite. More precisely, various representations although "copying" or "embodying" the same abstract ideas are often conflictual in that they refer to semantically different objects. The abstraction supposedly made easier by the multiple representations are double. On the one hand, one must see the "abstractness" into the "concreteness" via appropriate symbolism. On the other hand, one must also discover (on the long run) that the "abstractness" is semantically multiple. This paper will show that signed numbers can be interpreted as two semantically different sets of objects.

¹Groupe d'étude sur les problèmes de représentation en enseignement des mathématiques.

SONDER (1976) and FISCHBEIN (1977) pointed out various qualities of good models, questions which we also addressed in JANVIER (1980). We all criticize certain models on the basis that they engage the pupils in dealing with an interpretation which is not "natural". As SONDER puts it: "The representation should be mathematically honest and not phoney and contrived." FISCHBEIN explains: "... We very often use elaborated aids, such as graphs, schemas, structured materials, games. Such concrete models are useful as teaching aids if they have ... the capacity to stimulate the process of reasoning and to permit its progress by their own means¹ ... A model which is not internally structured, which exists only on the account of the original is only a pseudo-concrete model. It is in fact a system of symbols."

In other words, models must have some sort of internal life, must pre-exist in the mind of the pupil before he uses it. Some mental images concerning the situation must be available. In a way, the word "embodiment" by its meaning "to incorporate" exemplifies this criticism. Indeed, it stresses the fact that an abstract notion needs some concrete representation. However, the concreteness of the representation may stay at the level of colors, shapes, funny characters... The relations between those elements may be artificial, contrived and consequently quite abstract (even very abstract if it goes against children's common sense).

B) Models for signed or directed numbers.

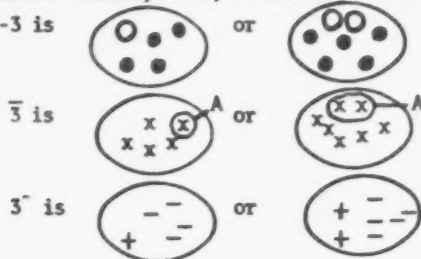
Apart from a few papers, most articles on this topic present a particular model and put forward some arguments to prove its excellence. A careful analysis of those models has shown that they can be subdivided into two main categories: the equilibrium model and the number line model.

a) The equilibrium model.

In this model, the numbers are elements of two "opposite" kinds:

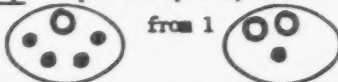
¹We underline.

black and white marbles, cheques and bills, negative and positive charges, outside and inside darts... Since adding those elements bring about the problem of dealing with mixture, it is possible to consider at the outset signed numbers as being a mixture of elements of opposite kinds. Clearly then, we introduce classes of equivalence. For example, -3 is

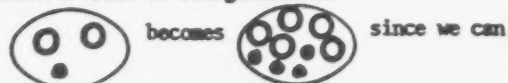


Adding is then defined as combining. Subtraction is defined as removing or taking away. A problem quickly arises:

Can we remove -3 from 1 since we do not



find enough black dots ● ? The question is solved in two ways. Firstly, we can introduce a "rule of reorganisation"



add any pair of opposite elements (this means that we can take any element of the equivalence class). Secondly, if this appears too complex we may simply define the subtraction as the addition of the opposite. But we then are far from the fundamental meaning. As for the multiplication, it can be defined as a repeated addition or subtraction. For example, $3 \times -4 = +(-4) + (-4) + (-4)$; $-3 \times -4 = -(-4) - (-4) - (-4)$. As we note, " $- \times -$ " is explained as $-(-n)$ for $n \in \mathbb{N}$ is.

Although coherent, experimentation has shown that pupils have problems with subtractions and even additions.

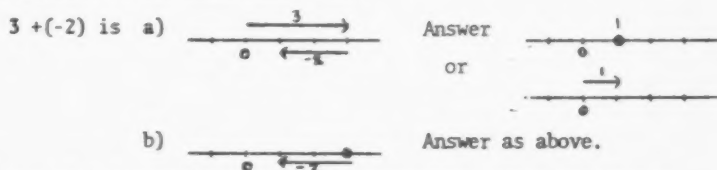
Symbolism can be introduced in different manners. Pupils can be introduced with some intermediate notation such as 5^0 (five out), 6^A (six against), and the signs "+" and "-" can be placed in several

positions.

Finally, this category of model could as be called the "ordered-pairs" category since it is essentially based on the idea of defining a signed number by a couple of positive numbers.

b) The number line model.

In this well-known model, numbers are either position on the number line or displacements on this line (from the origin or not). We see that ambiguities are introduced at the beginning. Addition then becomes a combination of these elements. Addition should work with objects of the same kind! Ex.:



The reader has certainly noted the continuous ambiguity between states and changes. The difficulty then arises from the fact that adding may be the composition of two changes or the change (transformation) from a state to a state.

Although children can learn a rule and follow it, this basic confusion creeps in when we introduce the subtraction. Indeed, "add the opposite" can be proposed or "turn around" before moving. In either case, the rule proves to be difficult for children. A lot has to be handled on the number line. In fact, it looks like if pupils who succeed pay not much attention to the model and make mostly computation. At any rate subtraction is still a complex operation. Multiplication is defined, as in the previous category, as repeated addition and subtraction.

c) A hybrid model.

In a recent work, Ahmed Daïf did with us, he discovered a model which belongs to each category¹. Indeed, it uses the number line and the idea of equilibrium. The signed numbers are represented by

¹Invented by LUTH (1967).

the actions of a "flying balloon" to which are attached bags or small balloons. Bags have a pulling-down effect, small balloons a rising effect. The effect is indicated by a label stuck to them. It is normally in equilibrium and once in a while a helicopter brings bags and small balloons. Addition is clearly identified with adding to the flying balloon, subtraction is defined as removing objects. We see that taking away a bag of 4 meters is equivalent to adding a small balloon of 4 meters.



C) An experiment.

A student of ours (A. Daïf) has tested this model (model A) against a model (model B) which used red-colored numbers for positive numbers and green-colored numbers for negatives ones. This model was called symbolic because the pupils mainly work with the symbols without referring to the initial situation. The hybrid model was called schematic because it presupposes some references to the mental images of the initial situation.

The model A group was composed of 76 pupils of grade 8 (second year of secondary level) and the control group (model B) contained 106 pupils of identical classes. The experiment was carried out in Morocco where the mathematics program is largely inspired by the official French program. No pre-tests were conducted. Tests were constructed on the basis of previous work by COTE (1981) so as to encompass all sorts of items. Table 1 shows the results obtained from each group on each item. As we see, each item is described by a code involving two signs and an inequality sign. The two signs refer to the sign of the operands, the inequality sign shows if the absolute value of the first operand is $>$, $=$ or $<$ than the absolute value of the second.

Ex.: $(-7) + (+4)$ would be coded - + >

$-3 + -4$ " - - <

The tests were double. Each item was reproduced using the symbolism of the model. For example, $(-7) + (+4)$ was presented as $7^s + 4^b$

(s: bag; b: balloon).

At the conference, we will discuss about the psychological peculiarities of the hybrid model and present more fully the results of the experiment.

Claude Janvier
UQAM
August 1983

Note: Bibliography provided at the conference.

TABLE 1

Success (%) in the arithmetic items

		Forme	Groupe A	Groupe B	Différence	P
A D D I T I O N	1	+ + >	94,7	87,6	7,1	0,1052
	4	- - >	87,5	67,6	19,9	0,0002
	5	- - =	79,6	60,4	19,2	0,0065
	6	- - <	88,2	61,9	26,3	0,0001
	7	+ - >	88,8	80,0	8,8	0,2065
	8	+ - =	90,8	77,1	13,7	0,0162
	9	+ - <	82,9	66,2	16,7	0,0214
	10	- + >	92,1	79,0	13,1	0,0165
S O U S T R A C T I O N	12	- + <	94,7	69,5	25,2	0,0000
	17	+ + >	89,5	87,6	1,9	0,7006
	18	+ + =	89,5	84,8	4,7	0,3563
	20	- - >	77,2	85,7	- 8,5	0,0890
	21	- - =	85,5	87,6	- 2,1	0,6820
	22	- - <	78,9	73,8	5,1	0,5816
	23	+ - >	73,7	73,3	0,4	0,8451
	24	+ - =	72,4	68,6	3,8	0,5816
	25	+ - <	70,4	71,6	- 1,2	0,6590
	26	- + >	72,4	60,0	12,4	0,0420
	27	- + =	63,1	51,4	11,7	0,1166
	32	0 -	56,6	70,9	-14,3	0,0903

A CONCEPT MAP FOR FRACTION MULTIPLICATION

Harold W. Mick

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Cette étude décrit un modèle qui met en lumière les relations (en gardant les structures) entre une structure de surface (système symbolique) et les structures profondes (significations conceptuelles) pour créer un plan conceptuel pour la multiplication des fractions. Le plan conceptuel représente la multiplication de fractions comme une accumulation de possibilités de solutions qui forment des structures conceptuelles. Pour éprouver le plan, on a observé comment quatre adolescents ont cherché à résoudre des problèmes.

For several years, I have been assessing adolescents' knowledge of fractions operations, particularly fraction multiplication. My principal reason for doing this is to confirm my conjecture that students merely shove numbers around without understanding the supporting mathematical meaning. In one recent study, I interviewed four students as they worked out problems related to fraction multiplication. From the tape transcripts I received a wealth of data, consisting of their responses as I constructed and explored hypotheses about their mental processes as they worked. It seemed clear to me that the subjects were mostly working with their minds in "neutral" -- but not entirely; they also possessed traces of conceptual knowledge, the sort you might expect in the early stages of concepts formation. But I lacked an organizing scheme that clearly showed how symbols and their meanings interacted during the problem solving process. I needed a "concept map," I reasoned, a type of map that would represent the important symbols, conceptual meanings, and their interrelationships for fraction multiplication.

In my search, I found several sources that used promising representational schemes. Wickelgren (1974) describes a problem solving model based on information processing theories. He defines a solution to a

problem as:

...an ordered succession or sequence of problem states, starting with the given state and terminating with a goal state, such that each successive state is obtained from the preceding state by means of an allowable action (operation applied to one or more expressions in the preceding state). (p. 16)

However, Wickelgren's "state-action tree" diagrams of all solutions to a single problem, are not maps but combinatorial trees showing all possible solutions. In his scheme, "a node [point] represents the sequence of actions [lines] or the sequence of states that led up to it, not the problem state achieved by that sequence of actions or states" (Wickelgren, p. 18). Such a tree differs from what I sought, namely a topological map, where each point represents a unique state, making it possible to show solutions to several problems on a single map. Skemp (1979) describes a goal seeking (problem solving) theory that uses topological maps. In contrast to a static model like Wickelgren's, Skemp's shows the dynamics of knowledge (conceptual structures) by representing solutions as paths within the conceptual structures. For Skemp, conceptual structures are networks of concepts (states of knowledge) plus connecting relations (transformations). These structures are represented as topological maps of points (states) and directed lines (transformations). Briefly, a problem's given information is first placed in an appropriate conceptual structure. Then through a sequence of actions (transformations), changes of knowledge states take place change by change in the course of moving through the network until they achieve a goal state. As in Wickelgren's model, the solution is a sequence of states and transformations; the difference is that the solution path is plotted within a conceptual structure appropriate to the problem. Burch and Brazier (1982), adapting Skemp's sequence, developed a procedure to determine representations of conceptual structures in trigonometry. They found both a problem solving structure and a condensed parent structure, and represented them with topological maps.

I used a procedure adapted from Burch and Brazier (1982) to construct conceptual structures of fraction multiplication. But unlike Burch and Brazier (1982), and Skemp (1979), I separated symbols from meanings (Skemp, 1982). This adaptation allowed me to highlight the structure-preserving relations between surface and deep structures, to create what I call a "concept map."

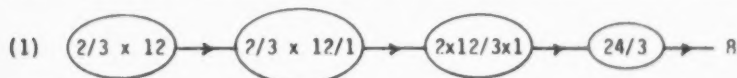
This paper has several purposes: (1) to show that the Burch and Brazier procedure, with some adaptations, can be applied to fraction multiplication, (2) to develop a concept map for fraction multiplication, (3) to show that the map has application for assessing and diagnosing fraction multiplication knowledge, and (4) to generalize the procedure for assessment beyond fraction multiplication.

METHOD

Imagine, for a moment, your home and a neighborhood grocery store. Not only can you picture these places in your mind, but you can also journey from one to the other. In effect, you have a neighborhood map stored in your mind, and you use this internal map to guide you when you go to and from the store. Now if an observer wanted to assess your knowledge of getting from your home to the grocery store, he might ask to go along and record your route on a neighborhood map by tracing boldly over the streets you traveled.

Now imagine, for a moment, your knowledge of fraction multiplication. What images come to mind? Of course, it is more difficult to picture fraction multiplication than to picture your neighborhood, because multiplication is more abstract. But you have an "internal map" for fraction multiplication to guide you when you multiply fractions, just as your internal map of the neighborhood guides you to the grocery store. Now suppose that same observer asks you to multiply $2/3 \times 12$. You comply and write $2/3 \times 12 = 2/3 \times 12/1 = 2 \times 12/3 \times 1 =$

$24/3 = 8$. Then, in a way similar to the way he traced the route from your home to the grocery store, the observer traces out this path:



Each ellipse represents a concept or state of knowledge of fraction multiplication just as a point represents a location in physical space on your neighborhood map. For example, the initial point " $2/3 \times 12$ " represents the given information of the problem, and the terminal point " 8 " represents the problem's answer. The arrows represent transformations that change one state of knowledge to another just as your neighborhood roads provide you with the means to change from one location to another. In short, the entire path represents your approach to the solution, just as the boldly traced path represented your route to the store.

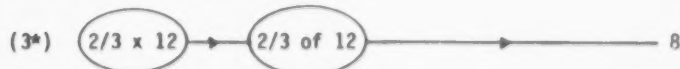
Now very briefly, I shall present four more solutions to " $2/3 \times 12$." The second solution is also one that stays within the symbol system:



For the third solution, interpret the $2/3$ as an operator and the 12 as a state of quantity (e.g., 12 marbles), and apply the $2/3$ operator to 12 by first partitioning 12 into three equal sized groups of marbles, and then combining two of the groups to get eight marbles.

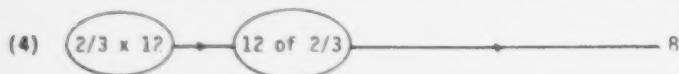


I shall condense this solution path and others within deep structures to the following simplified format:

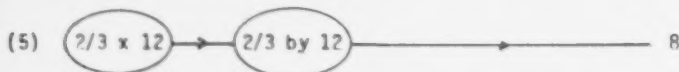


For the fourth solution, interpret 12 as the operator and consider $2/3$ to be a state of quantity (e.g., a portion of pie), and apply the 12 operator to $2/3$ by considering 12 portions each of $2/3$ quantity

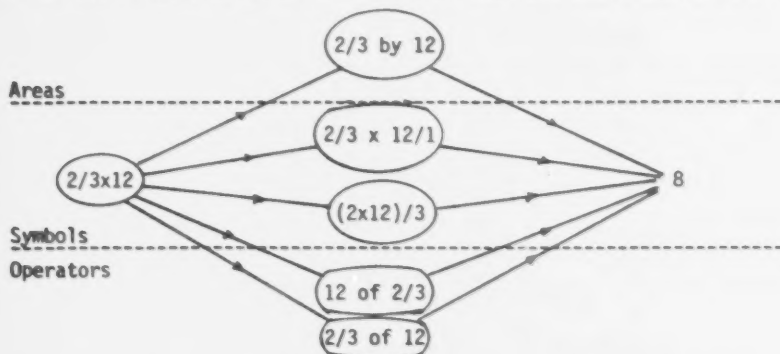
giving 12 of $2/3$'s pieces of pie, which is equivalent to 8 pies.



And finally, interpret both factors as linear measures and form a rectangle with dimensions $2/3$ and 12 units. The eight square units of area represent the product.



These five solutions go together to form the following concept map:



Note that in the above map each ellipse represents not the single state listed but a structure with further interiority of states and relations, in the same way as a dot on a globe can represent the entire city of New York (Skemp, 1982, p. 285). Also note that the sections labeled "Areas" and "Operators" are deep structures, whereas the section labeled "Symbols" is the surface structure. Since the two deep structures were found by interpreting the individual factors, you might expect that other deep structures will be found as other interpretations are made, e.g., ratio and division; however, ratios and divisions are not sensible interpretations for fraction multiplication, and therefore I found no other deep structures. The arrows crossing the dotted lines represent the structure preserving relations between surface and deep structures, while the other arrows represent within-structure relations.

Next, I chose problems of the form $a/b \times c/d$ and worked out solutions. And finally I decided to look at problems of the form $m \times n$, where m and n are whole numbers. Since fraction multiplication can be viewed as a mathematical generalization of whole number multiplication, it seemed appropriate to assess how well students make this generalization. This step completed the first draft of the concept map.

Next, I took the excerpts on fraction multiplication from transcripts of interviews with the two fifth graders and two sixth graders that I mentioned at the beginning of the paper. During this step I went through each transcript carefully, recording each sequence of states and transformations the subjects used. I then compared these sequences with the theoretical structure I had constructed. As a result, I made some minor changes in the states, transformations, and organization of the theoretical structure.

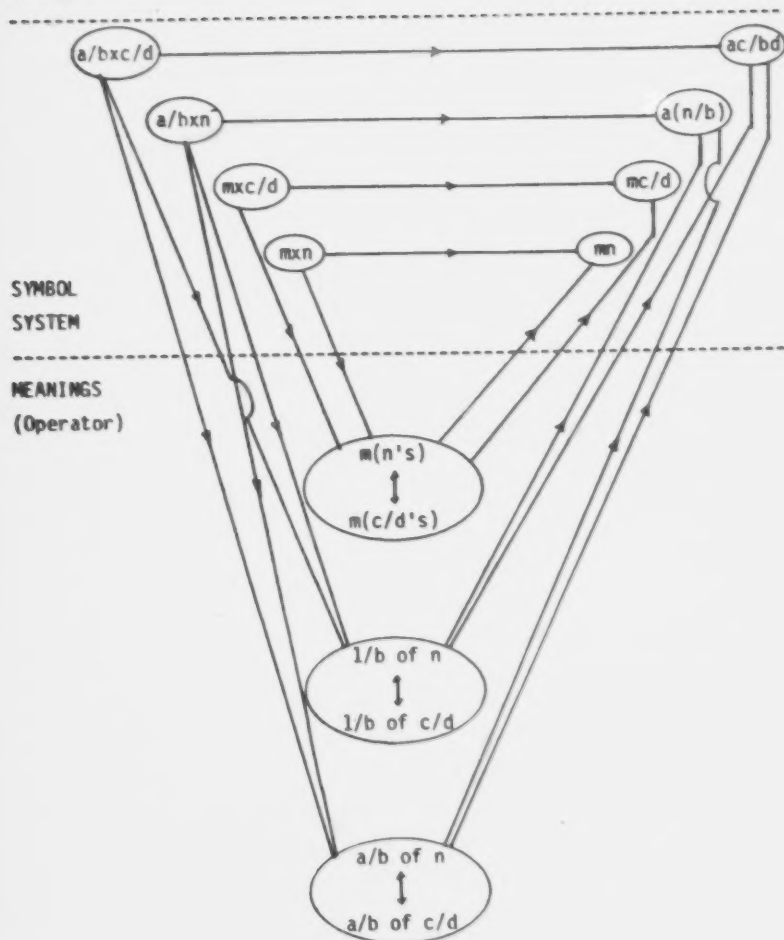
Now I shall generalize the steps used in constructing a concept map: (1) find appropriate problems representing the topic being mapped, (2) solve each problem in as many ways as possible, (3) sketch solutions paths for each solution (it helps to work out all solutions to a single problem before moving to another problem), (4) identify individual states of knowledge and transformations, (5) put the solutions together to form a concept map, (6) test the map by assessing subjects' knowledge of the topic, and (7) make needed revisions or additions in the map sketched in step (5).

Step (5) is the most difficult. Sometimes a map may go through several revisions before it represents (or expresses) the "best" topological fit, and sometimes intermediate or "missing" states of knowledge or transformations are hard to infer and characterize. My final version of the concept map for fraction multiplication is found on the next page.

Rectangular
Area

MEANINGS

(Area)



Concept Map for Fraction Multiplication.

DISCUSSION

While the initial task of constructing the theoretical structure was formidable, once formed the structure can be used over and over for assessment and diagnosis, and probably will pass through little change over time and use, though some further refinement should be expected. At this time, however, the complexity of the process appears to be a weakness, and it appears to be equally difficult to apply it to other topics. Also, it needs to be determined whether consistency of transcript evaluations can be obtained by different evaluators. In reflecting on my experiences with the four subjects, I find that more judgment entered into the assessment process than would normally be the case since the interviews were not originally designed around the proposed model. Nevertheless I found the interviews to be enormously useful in my analysis of the model, and eventually I managed to obtain a high degree of self-consistency in constructing and sketching the subjects' knowledge of fraction multiplication.

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PROBABILITY IN THE MIDDLE GRADES

Zacchaeus Kunle Oguntebi, Glenda Lappan, Michigan State University

RESUME

La recherche que nous présentons est une partie de l'évaluation du "Middle Grades Mathematic Project". (Le rapport "Teaching Similarity Concepts in the Middle Grades", présenté à cette conférence par Friedlander et Fitzgerald, explique le MGMP et sa philosophie.) Pendant l'automne de 1982, 30 professeurs ont employé l'unité de probabilité du MGMP avec un total de 1460 étudiants âgés de 11, 12, et 13 ans. Les classes représentaient un éventail de milieux (urbains, de banlieue, ruraux). Pour tous les étudiants, il y avait des examens avant et après l'enseignement avec l'unité de probabilité. L'analyse des résultats des examens, indique que tous les étudiants ont amélioré significativement leur performance suite à l'enseignement avec l'unité de probabilité. L'âge de 12 ans semble le meilleur âge pour une amélioration des résultats. Les notes des étudiants de milieux urbains ont été moindres que celles des autres étudiants et ce, pour les deux examens, mais ils ont amélioré leur performance autant que les autres. Dans les milieux ruraux, les garçons de 11 ans ont eu des notes plus élevées que les filles du même âge avant l'enseignement, mais après l'enseignement il n'y avait plus de différence entre les résultats des garçons et ceux des filles.

There is wide spread agreement that the middle grades 6 through 8 are crucial grades in the mathematical education of children and yet, these are precisely the grades where mathematics instruction is most weak and where the curriculum is least adequate. In response to this concern the Middle Grades Mathematics Project, funded by NSF-DISE, has developed and evaluated four units of high quality curriculum material. Because the project staff believes that the

developmental needs of early adolescents is best met by active learning experiences, detailed teacher guides were written for each unit in a format which enhances the teachers' ability to teach the content in an activity centered, problem solving mode. This paper reports the evaluation of the NCMP Probability unit.

Probability theory is the underpinning of the modern world. Current research in both the physical and social sciences cannot be understood without it. Today's politics, tomorrow's weather, and next weeks' satellite all depend on it.

This statement, Huff and Geis (1959), emphasizes the importance of understanding probabilistic notions, Shulte (1981, ix) asserts:

All major curriculum groups in this century - - including the NCTM in its recommendations for the curriculum of the 1980's -- have stressed the importance of statistics and probability...

If probability holds this place of importance in society then research and development efforts in teaching children probability are timely.

The objective of this study is to consider questions that will have implications for teaching probability at the middle grades level. For example, what is the level of understanding of middle grades students in probability prior to any curriculum intervention? Leake, Jr. (1965), and Doherty (1965) , who conducted their studies respectively on grade levels (7-9) and (4-6) conclude that elementary and junior high school boys and girls in general possess considerable knowledge of some probability concepts prior to formal instruction.

The literature includes studies that support the middle grades as an appropriate level for probability instruction. Armstrong (1972) investigated the ability of fifth and sixth graders. He found that sixth graders possessed the ability to learn all the concepts of probability taught; fifth graders were unable to learn the concept of outcome space. In another study, Smith (1966) developed and taught

a unit on probability and statistics for seventeen days to three groups of seventh graders. Smith concluded that the seventh grade was an appropriate level to introduce "at least some topics in probability and statistics."

Another point of interest is the question of sex differences in achievement in probability. Studies by Mullenex (1968), Doherty (1965), Smith (1966), and Wavering (1979), conducted respectively with grades levels 3-6, 4-6, 7, and (8, 10, and 12) and in varied settings, reported no sex differences. In an evaluation of the Comprehensive School Mathematics Project (CSMP) probability curriculum, sex differences were reported prior to instruction in both CSMP and non-CSMP students. However, Dougherty reported at the NCTM 1981 St. Louis meeting that after instruction sex differences were not found with CSMP students, but persisted with non-CSMP students. Kass (1964), reported that boys outperformed girls in binomial probability tasks.

PURPOSE

The main goal of the study was to evaluate the Probability Unit's effect on students and teachers. In order to accomplish this goal, the study first determined any existing differences in probability abilities and in attitudes toward mathematics of students in grades six through eight by sex and by grade level for each site prior to any instructional intervention. The second stage was to analyze the effects of instruction on skill development in probability and on attitudes toward mathematics and toward the unit again by sex and by grade for each site.

METHODOLOGY

Data was collected between October 1982 and January 1983 and included general information on the populations in each site, pre and post test scores from the subjects on a probability test and an attitude scale, and opinionnaire data from both students and teachers. The

subjects were sixth, seventh, and eight grade students from six different schools, situated in three distinct sites, two schools per site. The three sites are categorized as urban, suburban and rural. Because of space limitations, this paper will report on the urban and suburban sites. The total sample comprises approximately 1460 subjects from 66 classes taught by 30 different teachers. Table 1 gives the distribution of the sample by grade, by sex, by site.

Grade	Urban			Table 1 Suburban			Rural		
	Boys	Girls	Total	Boys	Girls	Total	Boys	Girls	Total
6	76	73	149	162	178	340	70	63	133
7	66	72	138	53	48	101	65	75	140
8	74	83	157	108	78	180	57	51	108

The instrument used to measure student attitudes, was developed by Shumway and White (1981). The Probability Unit Test, MGNPPT, was developed by the staff. The Cronbach reliability coefficients for MAS ranged from 0.80 to 0.89. The MGNPPT 's ranged from 0.58 to 0.83, on the post test with most in the 70's and 80's.

The Probability Unit includes ten activity lessons requiring approximately three weeks of instruction. The activities focus on fair and unfair game situations, surveys, area models for compound events, expected value, binomial distributions, and applications.

The teachers involved in the study were given two 1 1/2 hour workshops and were provided with all instructional materials and equipment. The tests and attitude scales were administered by the teachers during the regular school day. The statistical analyses included multivariate and univariate analysis of variance and repeated measures.

RESULTS AND CONCLUSIONS

Table 2 includes pretest and post test means of the Probability

Test (PT, range 0-25) and the Mathematics Attitude Scale (MAS, range 1-5) scores for the urban and rural settings by grade and by sex. Table 3 provides the P values obtained from the Multivariate and Univariate analysis of Variance prior to instruction.

Table 2

		URBAN					RURAL				
		PT			MAS		PT			MAS	
		Pre N	Post M	Post M	Pre M	Post M	Pre N	Post M	Post M	Pre M	Post M
Grade 6	149	6.63	9.79	3.61	3.26	133	8.64	11.50	3.45	3.50	
Boys	76	6.97	10.16	3.57	3.24	70	9.26	12.06	3.42	3.50	
Girls	73	6.27	9.41	3.65	3.28	63	7.95	10.87	2.48	3.49	
Grade 7	138	7.62	11.09	3.46	3.38	140	9.13	13.98	3.21	3.10	
Boys	66	7.46	10.34	3.55	3.41	65	9.40	14.43	3.18	3.09	
Girls	72	7.78	11.75	3.37	3.35	75	8.89	13.59	3.23	3.11	
Grade 8	157	7.92	11.25	3.46	3.42	108	9.99	11.69	3.01	2.84	
Boys	74	8.16	11.91	3.54	3.45	57	10.26	12.04	2.84	2.77	
Girls	83	7.70	10.66	3.40	3.38	51	9.69	11.32	3.19	2.93	
Total	444	7.39	19.71	3.51	3.35	381	9.20	12.47	3.24	3.16	
Boys	216	7.53	10.82	3.55	3.36	192	9.60	12.85	3.17	3.14	
Girls	228	7.27	10.61	3.47	3.34	189	8.79	12.07	3.30	3.19	

Table 3

MULTIVARIATE AND UNIVARIATE ANALYSIS OF VARIANCE--
P VALUES FOR INTERACTIONS, SEX DIFFERENCES, AND GRADE
PLANNED COMPARISONS PRIOR TO INSTRUCTION

Source of Variation	D.F.	URBAN			RURAL		
		Multivariate	Univariate		Multivariate	Univariate	
		P	PT P	MAS P	P	PT P	MAS P
Grade							
G ₁ = 6 vs 7	1	.013	.019	.149	.005	.149	.010
G ₂ = (6,7)							
vs 8	1	.052	.029	.430	.0001	.002	.0004
Sex	1	.518	.389	.385	.002	.009	.083
Grade by Sex	2	.456	.465	.426	.438	.492	.289

PRIOR TO INSTRUCTION

1. In both urban and rural settings, boys and girls did not differ in their attitudes toward mathematics. This was also the case for the suburban setting.
2. In probability knowledge, sex differences were not found in the urban setting, but grade 6 boys outperformed grade 6 girls in the rural setting. In general, boys tended to slightly out score girls.
3. Probability knowledge increased with grade level (age) while attitudes toward mathematics decreased with grade level in both settings. There were, however, slight variations.

Table 4 gives the P values for the Multivariate and Univariate Analysis of Repeated Measures for the pre-post comparisons. Figure 1 shows the profiles of the Probability Test mean gains and averages for pre-post test by grade and by sex. In the table and figure, the upper and lower parts show the averages and gains respectively.

Figure 1 Profiles of PT Mean Gains and Averages

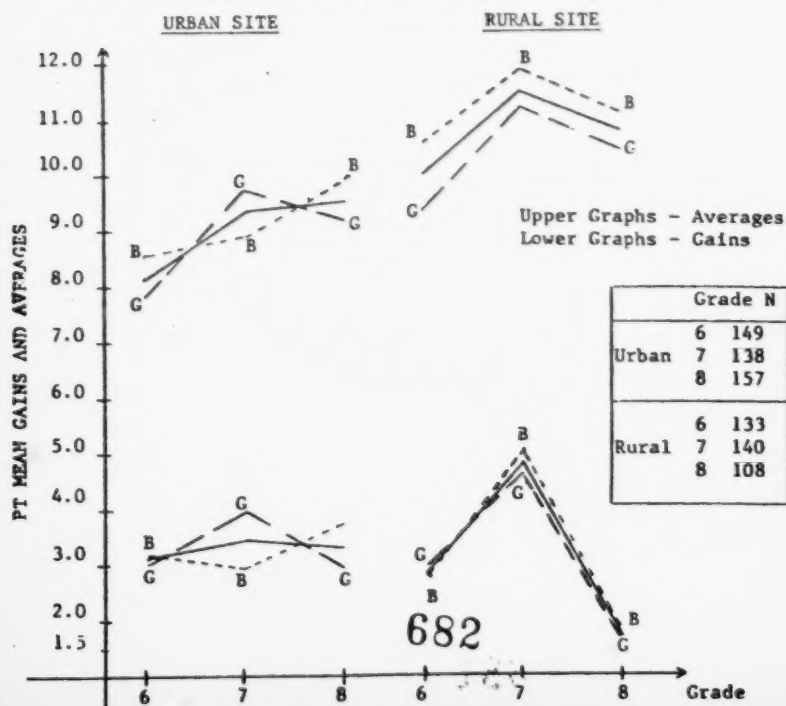


Table 4
ANALYSIS OF REPEATED MEASURES - P VALUES ONLY
FOR INTERACTIONS, SEX DIFFERENCES AND PLANNED COMPARISONS BY TIME

Source of Variation	Grade	URBAN			RURAL		
		Multivariate		Univariate	Multivariate		Univariate
		AVGPT* AVGMS*			AVGPT* AVGMS*		
		P	P	P	P	P	P
(6,7) vs 8	1	.862	.602	.802	.0001	.968	.0001
6 vs 7	1	.002	.001	.955	.0001	.0001	.0002
Sex	1	.657	.493	.470	.007	.010	.223
G by S	2	.166	.095	.55	.663	.668	.414
		DIFPT* DIFMS*			DIFPT* DIFMS*		
Time Effect	1	.0001	.0001	.0001	.0001	.0001	.033
G ₂ by T	1	.086	.951	.027	.0001	.0001	.073
G ₁ by T	1	.012	.425	.004	.0001	.0001	.052
S by T	1	.771	.894	.477	.348	.939	.152
G by S by T	2	.165	.057	.664	.876	.838	.661

*AVGPT = Averages of PT scores, AVGMS = Averages of MS scores.

*DIFPT = Differences in PT scores, DIFMS = Differences in MS scores

G = Grade, G₁ = 6 vs 7, G₂ = (6, 7) vs 8, T = Time, S = Sex

After Instruction

1. Middle grades (six, seven, and eight) boys and girls benefited significantly from the probability intervention. This means that the intervention works well for middle graders in any sociocultural setting.
2. In general, there was a slight but consistent decline in attitude toward mathematics. However, student responses to the opinionnaire tended to indicate otherwise.
3. Urban middle graders averaged (knew) less probability over the intervention period than their mates in the rural setting. Probability knowledge gains over the period were however the same
4. Those sex differences which were apparent prior to instruction disappeared after instruction. Hence boys and girls gained equally from the intervention, in spite of boys slight superiority in both pretest and post test scores. In the suburban setting (not emphasized in this report), girls slightly but consistently out gained boys in the intervention.
5. Grade seven seems to have gained more from the Probability instruction than grades six and eight. Hence grade 7 seems to be optional for the introduction of the Middle Grades Mathematics Project Probability Unit. On the whole, our findings showed that the Unit can work well for any of sixth, seventh, or eighth graders.

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SOME ASPECTS OF PROPORTIONAL REASONING IN YOUNG
CHILDREN

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RESUME

Cette étude considère quelques aspects du raisonnement proportionnel des enfants de 9 à 11 ans. Les problèmes choisis ont une structure mathématique simple (ne faisant intervenir que des divisions entières) et ont des contextes variés.

Les pourcentages de réussite sont assez élevés pour des enfants de l'âge considéré (entre 33 et 70%), et augmentent fortement entre 10 et 11 ans. Les problèmes comportant des mélanges sont plus difficiles que les autres. Le caractère familier d'un contexte ne facilite la solution que si les enfants utilisent couramment les proportions dans ce contexte.

De nombreuses stratégies "élémentaires", s'appuyant sur l'addition, sont observées. Parmi les facteurs cognitifs, le plus important semble être la capacité d'abstraction des effets de champ.

Le concept de proportion apparaît comme un concept multiforme dont le premier apprentissage est indépendant de la maîtrise des méthodes multiplicatives.

INTRODUCTION

Proportions (equalities of two ratios) are important both in everyday life -- in exchanges for instance -- and in the sciences. The large body of literature about proportional reasoning has described major strategies for solving proportion problems, and has reported on the influence of some task and student variables on performance.

A four-level scale of strategies has been proposed by Karplus (Karplus, Pulos & Stage, in press). At the lowest level, strategies are incomplete, i.e. do not use all the data. Consider the following problem:

Mike mixes 2 glasses of orange juice
and 6 glasses of water.
Now, he wants to make more orange juice,
of the SAME taste.
He uses 4 glasses of orange juice.
How many glasses of water should he use?

A subject answering "4, because there are 4 orange juice" uses an incomplete strategy. Qualitative strategies use all the data. Such a strategy is "10, because there is more orange juice than before, so I need more water too." At the next level of strategy, all the data is used and combined in an additive manner: "8, because $6 - 2 = 4$, and $4 + 4 = 8$." The highest level is that of multiplicative strategies. This classification is used and expanded in the study.

Many studies have investigated the influence of the number structure of the task on performance (Rupley, 1981; Noeiting, 1980). The presence of integer ratios and of units makes problem easier. Similarly, the difficulty increases with the size of the numbers and the ratios used.

This study concentrates on the following unanswered questions:
What are the patterns of development of proportional reasoning in young children? How does the problem context influence performance?

and What is the relationship between proportional reasoning and the ability to generate equivalent ratios? The subjects were chosen young (between 9 and 11 years of age). The number structure of the problems was kept at an elementary level by using small numbers and small integer ratios. On the other hand, the contexts of the problems were varied along two variables: familiarity and the presence of a mixture. Examples of familiar contexts are mixing paint and buying candy. An unfamiliar context is exchanging dollars for a mysterious currency, groats (referred to as the Dollars problem).

Method

The subjects were 60 third, fourth, and fifth graders from a middle-class, ethnically mixed elementary school. They were individually interviewed by the investigator. The interviews included two series construction problems -- designed to test the ability to generate equivalent ratios -- three proportion problems, and additional items not under discussion here. Group tests were also given, including measures of arithmetic proficiency, M-capacity, vocabulary, and field dependence-independence.

GENERAL RESULTS

Performance

The success rates for the problems of the study are presented in Table 1. They are quite high for such young subjects. Two explanations come to mind: first, the physical concepts used in the problems are simple, and second, the number structure of the problems is also elementary.

Table 1: Success Rates

	Grade			Total	
	3	4	5		
Series construction	40	40	68	49	N = 120
Proportion	47	52	72	57	N = 180

Comparing the performances at different grade levels, a large increase is visible between grades 4 and 5, while no major improvement can be seen between grades 3 and 4. That increase is due mainly to a decrease in the number of incomplete strategies.

Elementary multiplicative strategies

A number of correct, but non-traditional strategies was observed. Correct strategies are those methods which follow the logical structure of proportion (and would, if correctly followed, lead to the correct solution.) The classical correct method is the multiplicative method, in which the solution is reached via multiplications and divisions. A common strategy found in the study is the repeated addition strategy. On the example given above, the classical strategy is: "each orange juice uses 3 waters, so I need $4 \cdot 3 = 12$ waters." The repeated addition strategy is: "the first 2 orange juices use up 6 waters, and the other 2 use 6, so I need $6 + 6 = 12$ waters."

Why do subjects use repeated addition? Several explanations come to mind: subjects might use repeated addition because they do not feel comfortable using multiplications. They may do so because they do not fully comprehend the structure of proportion problems and believe the structure to be additive. Finally, the M-demand of multiplicative strategies may be higher than their M-capacity. An analysis of the influence of M-capacity on performance showed that subjects using repeated addition had lower M-capacity than those using multiplicative strategies, thereby supporting the last hypothesis. In order to distinguish between the first two, an analysis of the repeated adders' use of additive methods was made. It appeared that repeated adders are not more likely than others to use additive methods, whether on proportion or series construction problems. It is therefore reasonable to conclude that repeated adders understand the concept of proportion, but either do not feel comfortable with multiplication, or do not understand the link between repeated addition and multiplication. The concept of proportion can be understood without having the ability to use

strictly multiplicative methods on proportion problems.

Proportion and series construction

The success rates for proportion problems are higher than those for series construction problems for mixture problems (49 versus 33%), while the reverse is true for non-mixture problems (58 versus 76%). It therefore appears that the ability to generate equivalent ratios might precede proportional reasoning.

VARIABLES AFFECTING PERFORMANCE

Task variables

The study deliberately ignored the structural aspects of the task (i.e. the number structure of the problems) to concentrate on context variables. Two context variables will be discussed here: the familiarity of the context and the presence of mixtures.

A first remark about the familiarity of the context is that a functional definition is uneasy to give. Familiarity varies from Individual to individual, even within a given age group. Moreover, familiarity may not be the important variable for task difficulty. It appears that seemingly equally familiar contexts (mixing orange juice and paint) result in quite different success rates (60 and 37%). On the other hand, contexts that appear very different in their degree of familiarity (as in the Candy problem and the Dollars problem) have similar success rates (63 and 67%). Consequently, a more precise characteristic of the context, namely familiarity with the use of ratios in that context, seems to influence performance most. Going back to the examples given above, paint and orange juice are equally familiar to children of the age group considered, but orange juice is almost always associated with a (1:3) mixture with water, whereas most experiences with paint include no mixing, or only "qualitative" mixing, in which the colors are adjusted using trial and error. On the other hand, exchanges of money and goods are so familiar to children that they are not disturbed by the introduction of strange and unfamiliar words as in the Dollars

problem.

Presence of a mixture

Mixture problems appear to be much more difficult than non-mixture ones. This may be due to lack of familiarity with such contexts, but it is also associated with the difficulty to create a concrete representation for such problems (Pulos, Stage & Karplus, 1981).

Subject variables

Several cognitive characteristics were of interest in the study: M-capacity, general intelligence, field dependence-independence. A stepwise multiple regression between the various cognitive variables and the overall performance on proportion problems showed the main effect to be field dependence-independence. For series construction problems, the same method showed the main effect to be general intelligence.

CONCLUSION

Young children can solve simple proportion problems. They have some understanding of the concept of proportion before they master the multiplicative strategies.

Performance is better when the subject is familiar with the use of ratios in the context of the problem (familiarity with the context does not by itself insure good performance). The presence of a mixture makes problems more difficult.

Among the cognitive variables, field dependence-independence and general intelligence are important factors in the early development of proportional reasoning.

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